Skew-closed objects, typings of linear lambda terms, and flows on trivalent graphs

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[work in progress, in part based on joint work with Jason Reed]

Lambda calculus: linearity and related notions

a term is linear if every (free or bound) var is used exactly once

- linear: $\lambda x.\lambda y.xy$
- non-linear: $\lambda x.\lambda y.y$, $\lambda x.\lambda y.x(xy)$

a term is planar if variables are used in the order they're bound

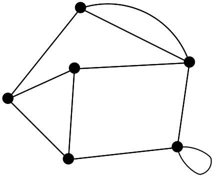
- planar: $\lambda x.\lambda y.\lambda z.x(yz)$
- non-planar: $\lambda x.\lambda y.\lambda z.(xz)y$

a term is unit-free if it has no closed subterms

- unit-free: $x \vdash \lambda y.yx$
- not unit-free: $x \vdash x(\lambda y.y)$

What are “maps”?

a graph + embedding into an oriented surface (e.g., the sphere)

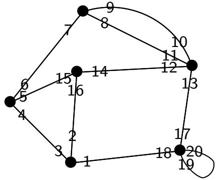


What are “maps”?

a graph + embedding into an oriented surface (e.g., the sphere)

or equivalently...

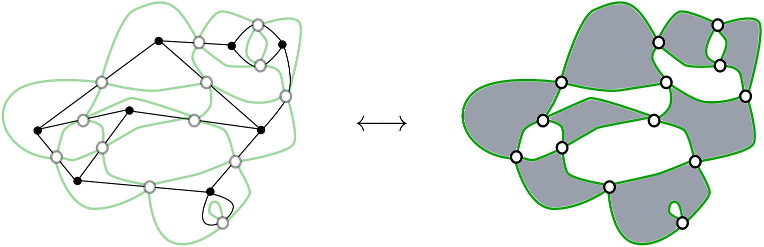
a permutation representation of $\Gamma = \langle v, e, f \mid e^2 = vef = 1 \rangle$



$v = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12\ 13)(14\ 15\ 16)(17\ 18\ 19\ 20)$
$e = (1\ 18)(2\ 16)(3\ 4)(5\ 15)(6\ 7)(8\ 11)(9\ 10)(12\ 14)(13\ 17)(19\ 20)$
$f = (1\ 17\ 12\ 16)(2\ 15\ 4)(3\ 6\ 9\ 13\ 20\ 18)(5\ 14\ 11\ 7)(8\ 10)(19)$

What are "maps"?

close connections to knot theory via the *medial map* construction[1]



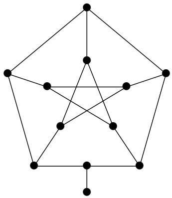
[1] cf. Louis Kauffman's "A Tutte polynomial for signed graphs", *Discrete Appl. Math.* 25 (1989), 105-127

What are "maps"?

Bill Tutte pioneered the enumerative study of maps.

- A census of planar triangulations. *Can. J. Math.* 14:21–38, 1962
- A census of Hamiltonian polygons. *Can. J. Math.* 14:402–417, 1962
- A census of planar maps. *Can. J. Math.* 15:249–271, 1963
- On the enumeration of planar maps. *Bull. AMS* 74:64–74, 1968
- On the enumeration of four-colored maps. *SIAM J. Appl. Math.* 17:454–460, 1969

One of Tutte's early insights was to consider *rooted* maps.



(a rooted trivalent map)

Some surprising enumerative connections

family of lambda terms	family of rooted maps	OEIS
linear terms[1,4]	trivalent maps	A062980
planar terms[4]	planar trivalent maps	A002005
unit-free linear[4]	bridgeless trivalent	A267827
unit-free planar[4]	bridgeless planar trivalent	A000309
normal linear terms/$\sim$[3]	maps	A000698
normal planar terms[2]	planar maps	A000168
normal unit-free linear/$\sim$[5]	bridgeless maps	A000699
normal unit-free planar[6]	bridgeless planar	A000260

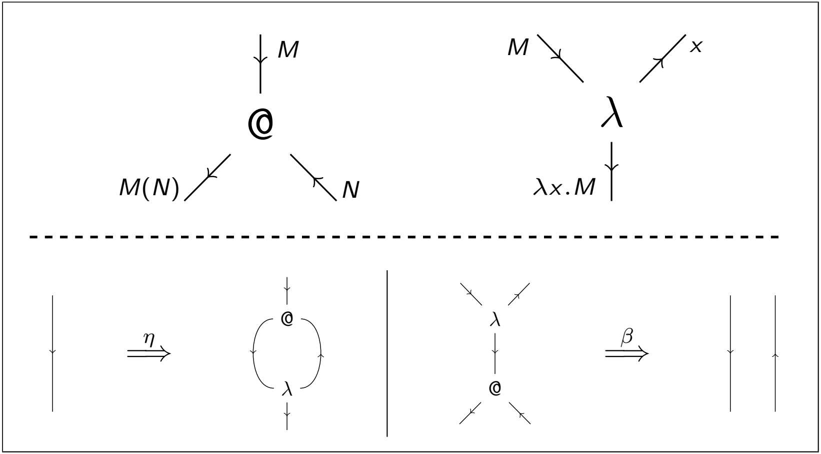
1. Bodini, Gardy, Jacquot, "Asymptotics and...", *TCS* 502, 2013.
2. Z, Giorgetti, "A correspondence between...", *LMCS* 11(3:22), 2015.
3. Z, "Counting isomorphism classes...", arXiv:1509.07596, 2015.
4. Z, "Linear lambda terms as invariants...", *JFP* 26(e21), 2016.
5. Courtiel, Yeats, Z, "Connected chord...", arXiv:1611.04611, 2017.
6. Z, "A sequent calculus for a semi-associative law", FSCD 2017.

String diagrams for linear lambda terms

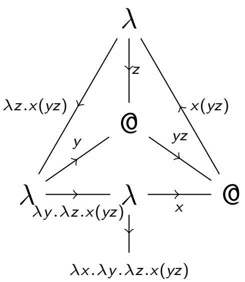
Linear lambda terms (with *n* free vars) may be modelled as (*n*-ary) endomorphisms of a **reflexive object** in a *symmetric monoidal closed bicategory*, i.e., an object U equipped with an adjunction $@ \dashv \lambda$ to its space of endomorphisms $U \multimap U$.

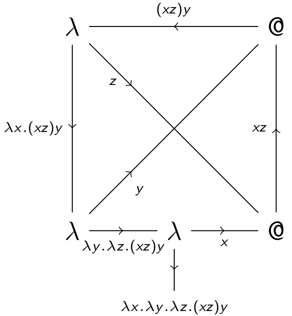
Interpreting this signature in the graphical language of *compact closed* bicategories ($U \multimap U \cong U \otimes U^*$) recovers a familiar diagrammatic notation for lambda terms...

String diagrams for linear lambda terms

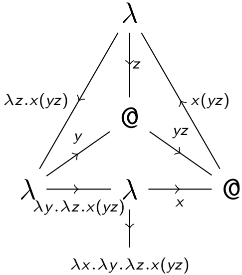


String diagrams for linear lambda terms

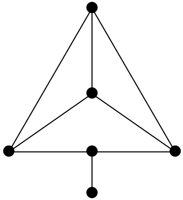




Linear lambda terms as invariants of rooted trivalent maps

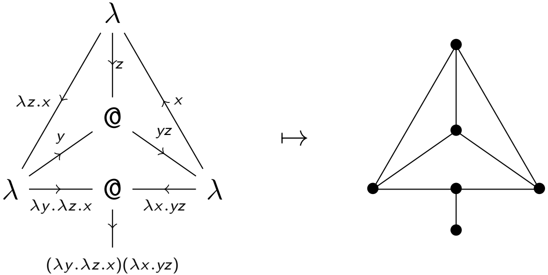


$\mapsto$



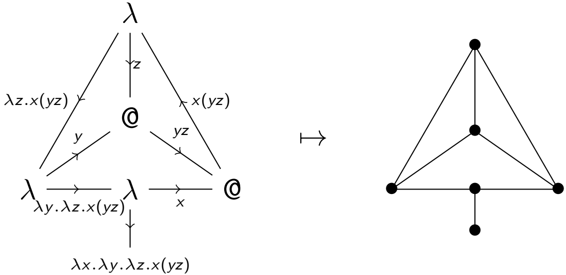
Forgetting edge orientations/vertex states yields a rooted map.

Linear lambda terms as invariants of rooted trivalent maps



Forgetting edge orientations/vertex states yields a rooted map.
...but not every orientation gives a valid lambda term!

Linear lambda terms as invariants of rooted trivalent maps



Forgetting edge orientations/vertex states yields a rooted map.
...but not every orientation gives a valid lambda term!

In fact, every rooted trivalent map is the underlying map of a *unique* linear lambda term. (Effectively, the term can be seen as a *complete topological invariant* of its underlying trivalent map.)

Typing as edge-coloring

So what about types?

Seen through the lens of graph theory, typing is naturally posed as an edge-coloring problem: assign each edge (= subterm) a color (= type) so as to satisfy certain constraints at the vertices (= applications and abstractions).

To make this analogy precise, let's first meet a friendly algebraic gadget...

Introducing imploids

An **imploid** is a preorder $(P, \leq)$ equipped with an operation

$$\frac{A_2 \leq A_1 \quad B_1 \leq B_2}{A_1 \multimap B_1 \leq A_2 \multimap B_2} \tag{1}$$

and an element $I \in P$, satisfying laws of *composition*, *identity*, *unit*:

$$B \multimap C \leq (A \multimap B) \multimap (A \multimap C) \tag{2}$$

$$I \leq A \multimap A \tag{3}$$

$$I \multimap A \leq A \tag{4}$$

In a **non-unital** imploid we only ask for (1) and (2). An imploid is said to be **commutative** if it moreover satisfies *DNI*:

$$A \leq (A \multimap B) \multimap B \tag{5}$$

Introducing imploids

Any group provides an example of an imploid, by taking the discrete preorder and $A \multimap B \stackrel{\text{def}}{=} B \bullet A^{-1}$.

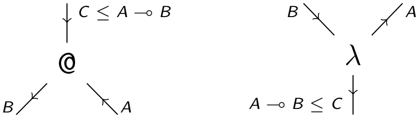
So does any (skew) monoid, by taking its downwards closed subsets ordered by inclusion and $A \multimap B \stackrel{\text{def}}{=} \{ x \mid \forall y.\, y \in A \Rightarrow x \bullet y \in B \}$.

Conversely, an imploid is just a **skew-closed** preorder:

- Ross Street. Skew-closed categories. *J. Pure and Appl. Alg.*, 217(6):973–988, 2013.

Imploid-typing

Let M be a linear lambda term, and $P = (P, \leq, \multimap, I)$ a commutative imploid. A P-**typing** of M is an assignment $\mathrm{Subterms}(M) \to P$ satisfying the constraints



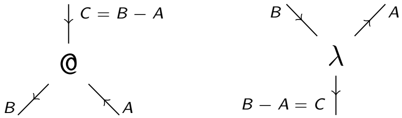
at every application and abstraction.

If M is planar we can drop assumption that P is commutative.

If M is unit-free we can drop assumption that P is unital.

Imploid-typings and G-flows

For $P = G$ a (commutative) group, a P-typing of M is the same thing as a *G-flow*[2] on its underlying trivalent graph $|M|$.

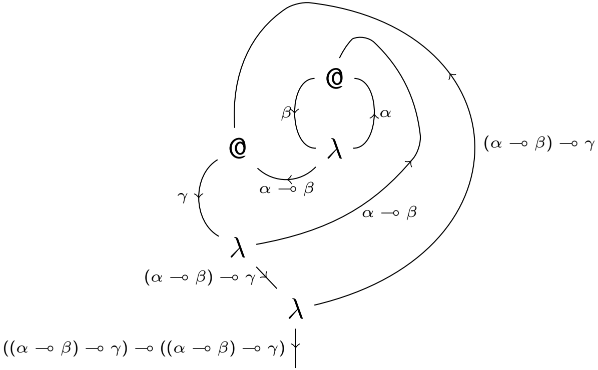


(sum of outputs = sum of inputs)

For example, a $\mathbb{Z}_2$-typing is the same thing as an element of the **cycle space** of $|M|$...

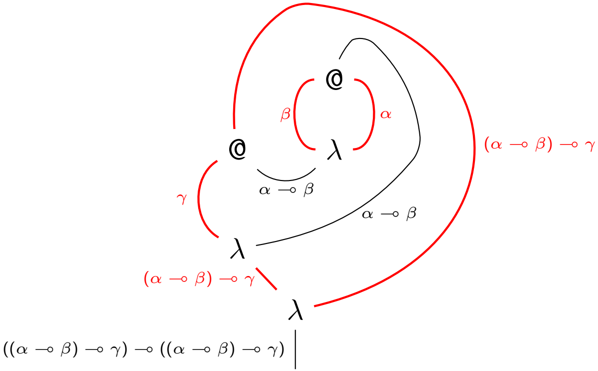
[2] W. T. Tutte, A contribution to the theory of chromatic polynomials. *Can. J. Math.* 6:80–91, 1954.

Imploid-typings and G-flows



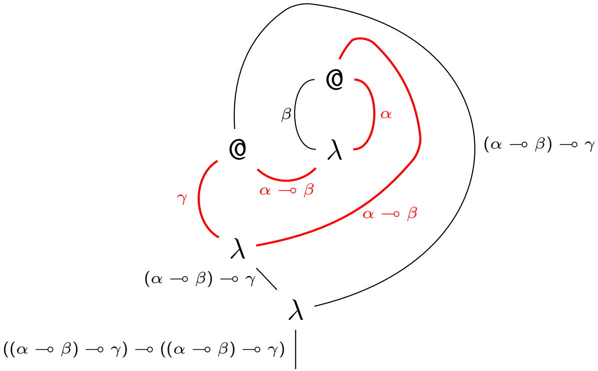
A Free(3)-typing

Imploid-typings and G-flows



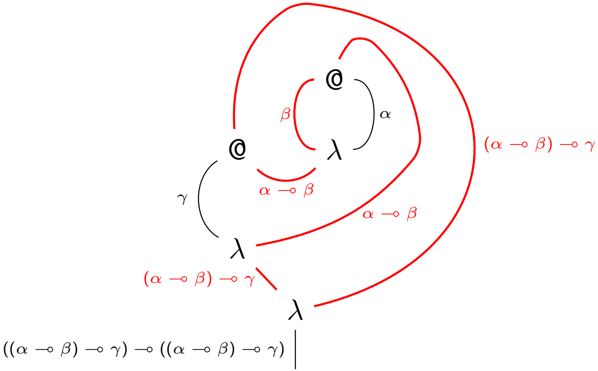
A $\mathbb{Z}_2$-typing with $\alpha = 1$, $\beta = 1$, $\gamma = 1$

Imploid-typings and G-flows



A $\mathbb{Z}_2$-typing with $\alpha = 1$, $\beta = 0$, $\gamma = 1$

Imploid-typings and G-flows



A $\mathbb{Z}_2$-typing with $\alpha = 0$, $\beta = 1$, $\gamma = 0$

The typing problem for linear lambda terms is usually considered "trivial", but the study of flows on graphs (and trivalent graphs in particular) is a deep and richly developed subject.

Let us say that a P-typing is **proper** if no subterm is assigned a type above the unit type I.

Theorem

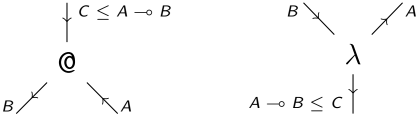
Every unit-free planar lambda term has a proper $\mathbb{Z}_2 \times \mathbb{Z}_2$-typing.

Proof.

This is equivalent to the Four Color Theorem. $\square$

Diagrams for skew-closed objects

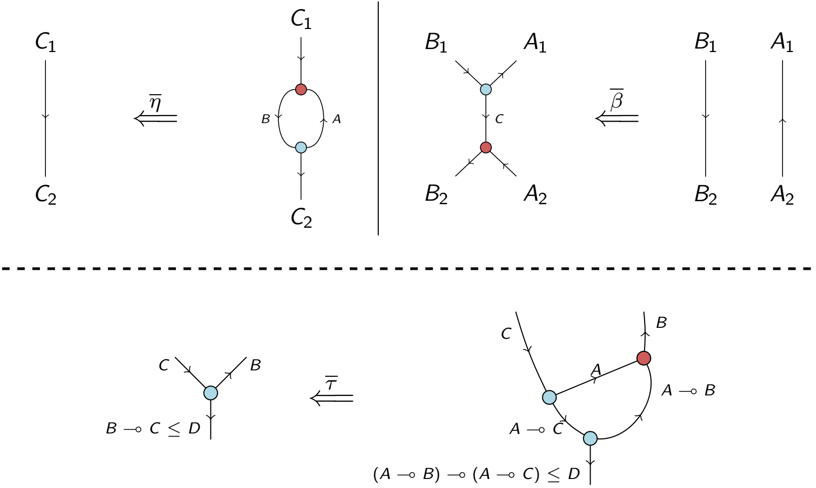
We can view the typing constraints



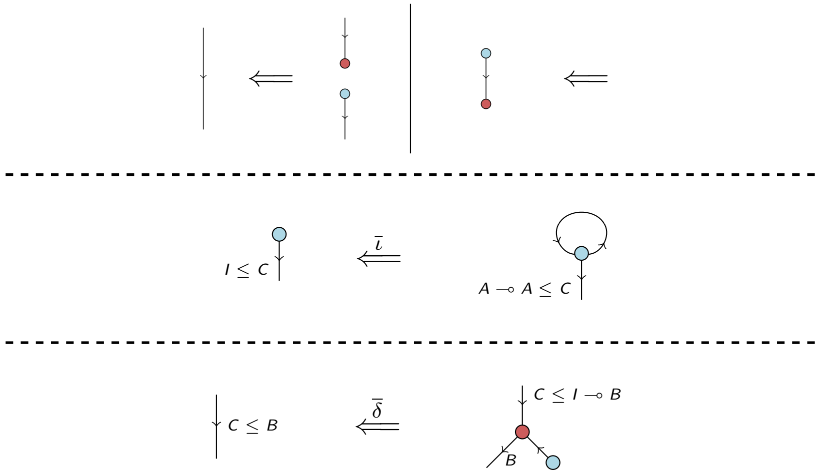
as defining (2-enriched) *distributors* $@ : P \nrightarrow P \otimes P^*$ and $\lambda : P \otimes P^* \nrightarrow P$. Indeed, these are exactly the adjoint pair of distributors $\lambda \dashv @$ associated to the functor $\multimap : P^{\mathrm{op}} \times P \to P$.

The definition of an imploid can be recast in diagrammatic terms ($@ = \bullet$, $\lambda = \circ$)...

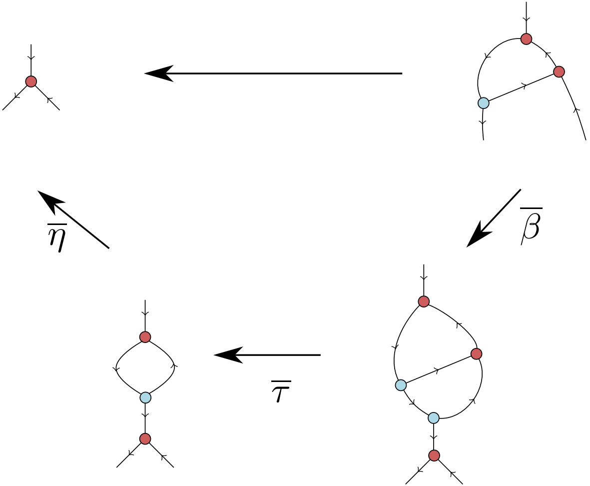
Diagrams for skew-closed objects (non-unital fragment)



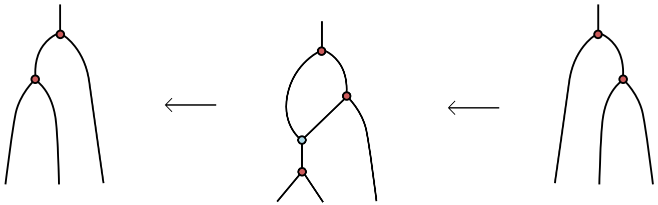
Diagrams for skew-closed objects (unital fragment)



Some derived rules



Some derived rules



A combinatory correctness criterion

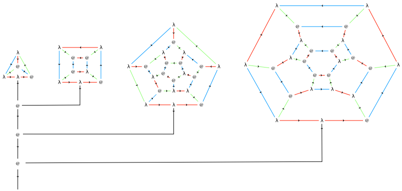
Theorem (Reed, Z)

A string diagram (with one incoming and one outgoing edge) represents a unit-free planar term $x \vdash M$ just in case it can be reduced to the trivial diagram $x \vdash x$ using only $\overline{\eta}$, $\overline{\beta}$, and $\overline{\tau}$ moves.

Proof.

($\Leftarrow$) is easy. ($\Rightarrow$) is by constructing a term $x \vdash T$ using only compositions of the "B" combinator $x \vdash \lambda y.\lambda z.x(yz)$ such that $T \to_\beta^* M$. □

The End?...



Questions:

- Coherence axioms on 2-cells?
- Any useful applications of combinatory completeness?
- Extension to D. Thurston's completeness theorem for the algebra of *knotted trivalent graphs*?
- How should we view the space of P-typings of a lambda term?
- Any meaning to flow/cut duality?