Tutorial: fibrational perspectives on logic and language

Part II: bifibrations

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RAMiCS 2026 @ Poznán, 8 April 2026

3. Bifibrations in logic and language

What is a bifibration?

Formally:

$$\begin{array}{c} \mathcal{E} \\ \pi \big\downarrow \\ \mathcal{B} \end{array}$$

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$$\begin{array}{ccccc} \mathcal{E} & & S & & \\ {\scriptstyle\pi}\big\downarrow & & \vdots & & \\ \mathcal{B} & & A & \xrightarrow{\;f\;} & B \end{array}$$

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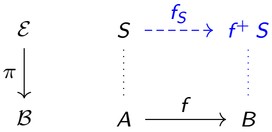

What is a bifibration?

Formally:

$$
\begin{array}{c}
\mathcal{E} \\
\pi \big\downarrow \\
\mathcal{B}
\end{array}
\qquad
\begin{array}{ccc}
S & \overset{f_S}{\dashrightarrow} & f^{+}\,S \\
\vdots & & \vdots \\
A & \xrightarrow{\quad f \quad} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & T \\
 & & \vdots \\
B & \xrightarrow{\quad g \quad} & C
\end{array}
$$

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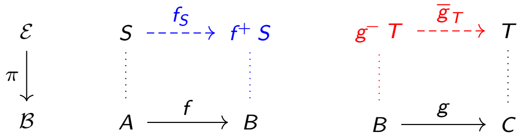

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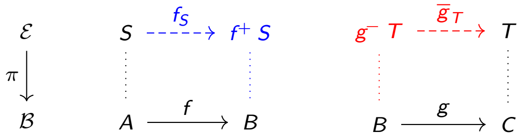

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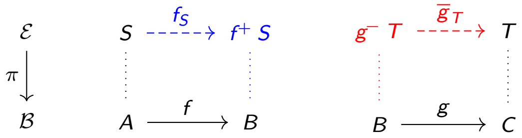

...and these liftings should be universal...

$$S \xrightarrow{\alpha} T$$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

What is a bifibration?

Formally:

$$
\begin{array}{c} \mathcal{E} \\ \pi \downarrow \\ \mathcal{B} \end{array}
\qquad
\begin{array}{ccc} S & \xrightarrow{f_S} & f^+ S \\ \vdots & & \vdots \\ A & \xrightarrow{f} & B \end{array}
\qquad
\begin{array}{ccc} g^- T & \xrightarrow{\overline{g}_T} & T \\ \vdots & & \vdots \\ B & \xrightarrow{g} & C \end{array}
$$

...and these liftings should be universal...

$$
\begin{array}{ccccc} S & \xrightarrow{f_S} & f^+ S & \xrightarrow{f \backslash_g \alpha} & T \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}
\quad = \quad
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...and these liftings should be universal...

$$
\begin{array}{c} S \xrightarrow{f_S} f^+ S \xrightarrow{f \backslash_g \alpha} T \\ \\ A \xrightarrow{f} B \xrightarrow{g} C \end{array}
=
\begin{array}{c} S \xrightarrow{\qquad \alpha \qquad} T \\ \\ A \xrightarrow{f} B \xrightarrow{g} C \end{array}
=
\begin{array}{c} S \xrightarrow{\alpha_f / \overline{g}} g^- T \xrightarrow{\overline{g}_T} T \\ \\ A \xrightarrow{f} B \xrightarrow{g} C \end{array}
$$

Bifibrations as families of adjunctions

A consequence of the definition is that in a bifibration $\pi : \mathcal{E} \to \mathcal{B}$, pushing and pulling along any given arrow $f : A \to B$ in $\mathcal{B}$ induces an *adjunction* between fiber categories:

$$\mathcal{E}_A \underset{f^-}{\overset{f^+}{\underset{\bot}{\rightleftarrows}}} \mathcal{E}_B \qquad\qquad \frac{S \Longrightarrow_{\mathrm{id}_A} f^- \, T}{f^+ \, S \Longrightarrow_{\mathrm{id}_B} T}$$

Indeed, any bifibration $\mathcal{E} \to \mathcal{B}$ can be transformed into a pseudofunctor $\mathcal{B} \to \mathrm{Adj}$, and vice versa. Thus bifibrations can be found everywhere there are adjunctions!

Example: bifibration of subsets over Set

Let Set be the category of sets and functions.

Let SubSet be the category whose objects are pairs $(A, S \subset A)$ and whose arrows $(A, S) \to (B, T)$ are functions $f : A \to B$ such that $x \in S \implies f(x) \in T$.

The forgetful functor $\mathrm{SubSet} \to \mathrm{Set}$ is a bifibration, where

$$f^{+}(A, S) = (B,\ f(S)\ )$$
$$f^{-}(B, T) = (A,\ f^{-1}(T)\ )$$

In particular, the adjunction $f^{+} \dashv f^{-}$ reduces to the fact that

$$f(S) \subseteq T \iff S \subseteq f^{-1}(T)$$

Example: bifibration of subsets over Rel

Let Rel be the category of sets and relations.

Let SubRel be the category whose objects are pairs $(A, S \subset A)$ and whose arrows $(A, S) \to (B, T)$ are relations $r : A \nrightarrow B$ such that $x \in S \wedge (x, y) \in r \implies y \in T$.

The forgetful functor $\mathrm{SubRel} \to \mathrm{Rel}$ is a bifibration, where

$$r^+(A, S) = (B,\ \{\, y \mid \exists x.(x, y) \in r \wedge x \in S \,\}\ )$$
$$r^-(B, T) = (A,\ \{\, x \mid \forall y.(x, y) \in r \supset y \in T \,\}\ )$$

(Compare $r^+ \dashv r^-$ with Galois connection $\blacklozenge\phi \vdash \psi \iff \phi \vdash \Box\psi$ from modal logic.)

Example: Hoare logic

Let Cmd be a one-object category whose arrows $c : * \to *$ are sequential compositions of program commands defined over some global set of variables.

Let Prf be the category whose objects are predicates on the global state, and whose arrows $P \to Q$ are given by commands c that satisfy the Hoare triple $\{P\}c\{Q\}$.

The following are equivalent:

1. the evident forgetful functor $\mathrm{Prf} \to \mathrm{Cmd}$ is a bifibration;
2. *weakest preconditions* and *strongest postconditions* exist for all P, c, Q.

$$P \Rightarrow wp(c, Q) \quad \Longleftrightarrow \quad \{P\}c\{Q\} \quad \Longleftrightarrow \quad sp(c, P) \Rightarrow Q$$

Example from language theory

Recall the poset of formal languages $(P(\Sigma^*), \subseteq)$ has the structure of a residuated monoid, with concatenation and left- and right-division of languages defined by:

$$K \cdot L \stackrel{\text{def}}{=} \{\, uv \mid u \in K \wedge v \in L \,\}$$
$$K \backslash L \stackrel{\text{def}}{=} \{\, v \mid u \in K \supset uv \in L \,\}$$
$$L / K \stackrel{\text{def}}{=} \{\, u \mid v \in K \supset uv \in L \,\}$$

This is an instance of the following general construction...

Example from language theory

Proposition. Let $(A, m : A \otimes A \to A, e : 1 \to A)$ be a monoid in the base of a symmetric monoidal closed bifibration $\pi : \mathcal{E} \to \mathcal{C}$. Then the fiber $\mathcal{E}_A$ is monoidal biclosed, with tensor and left- and right-internal homs defined by

$$S \otimes_A T \stackrel{\text{def}}{=} m^+(S \otimes_{\mathcal{E}} T) \qquad S \backslash_A T \stackrel{\text{def}}{=} m_\ell{}^-(S \multimap T) \qquad T /_A S \stackrel{\text{def}}{=} m_r{}^-(S \multimap T)$$

where $m_\ell, m_r : A \to A \multimap A$ denote the left- and right-curryings of m.

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Considering $(\Sigma^*, \cdot, \epsilon)$ as a monoid in the base of the cartesian closed bifibration $\mathrm{SubSet} \to \mathrm{Set}$ recovers the monoidal closed structure on $(P(\Sigma^*), \subseteq)$.

4. The logic of bifibrations

Free bifibrations

Most functors are not bifibrations, but one can try to freely turn them into bifibrations:

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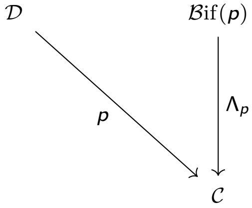

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$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\eta_p} & \mathcal{B}\mathrm{if}(p) \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \Lambda_p} \\ & & \mathcal{C} \end{array}$$

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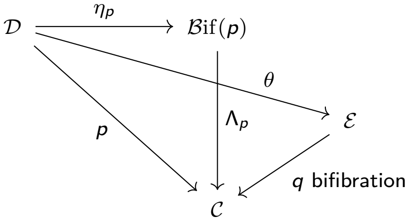

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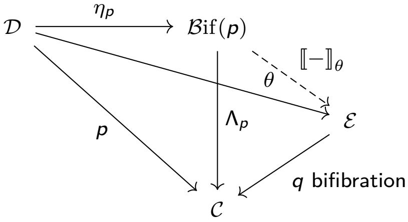

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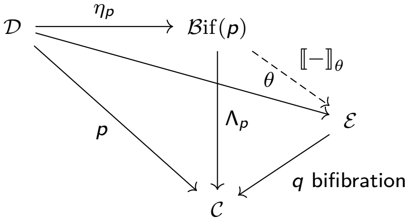

Surprisingly, this problem has been relatively little-studied:

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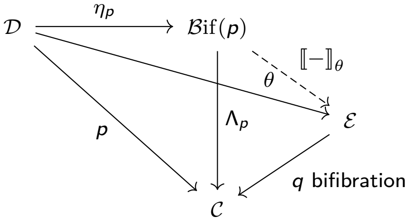

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In joint work with Bryce Clarke and Gabriel Scherer, we gave an explicit proof-theoretic construction and found some interesting examples of free bifibrations (arxiv:2511.07314).

Building the free bifibration: intuition

Introduce "formal" push/pull along the arrows of $\mathcal{C}$.

$$\mathsf{X} \xrightarrow{\quad \delta \in \mathcal{D} \quad} \mathsf{Y}$$

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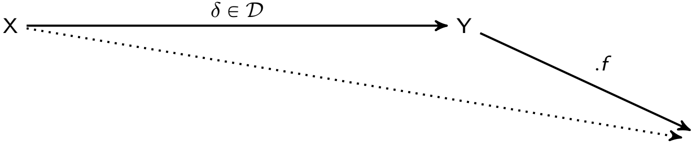

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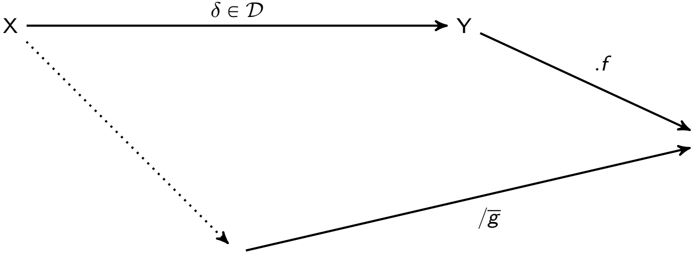

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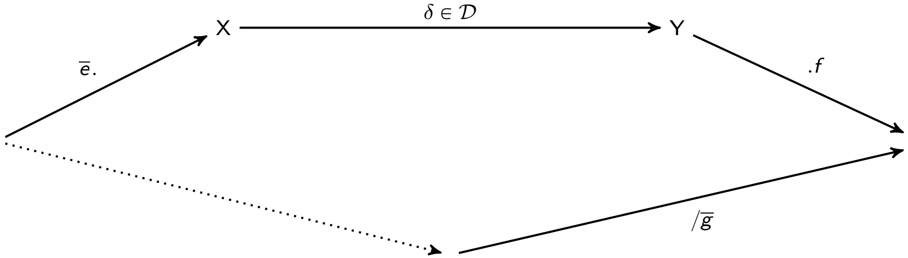

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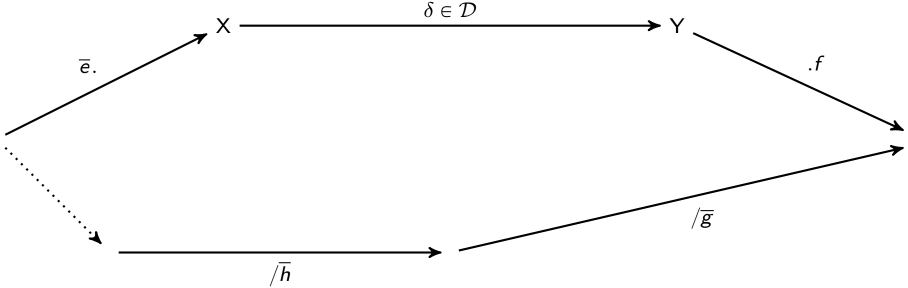

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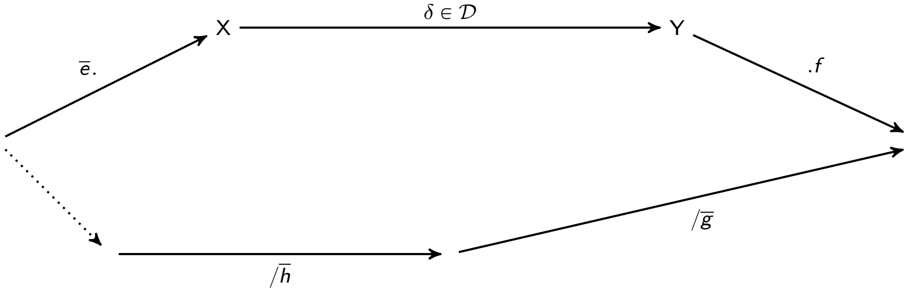
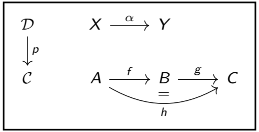


Some operations commute: non-trivial equivalence.

An example derivation in a free bifibration



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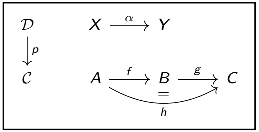

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \underset{f}{\Longrightarrow} Y}\,\alpha}{X \underset{fg}{\Longrightarrow} g^{+}\, Y}\,\mathsf{R}g^{+}}{f^{+}\, X \underset{g}{\Longrightarrow} g^{+}\, Y}\,\mathsf{L}f^{+}}{f^{-}\, f^{+}\, X \underset{fg}{\Longrightarrow} g^{+}\, Y}\,\mathsf{L}f^{-}}{f^{-}\, f^{+}\, X \underset{h}{\Longrightarrow} g^{+}\, Y}}{f^{-}\, f^{+}\, X \underset{\mathrm{id}_A}{\Longrightarrow} h^{-}\, g^{+}\, Y}\,\mathsf{R}h^{-}$$

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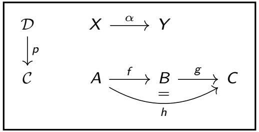


$$\dfrac{}{X \underset{f}{\Longrightarrow} Y}\ \alpha$$

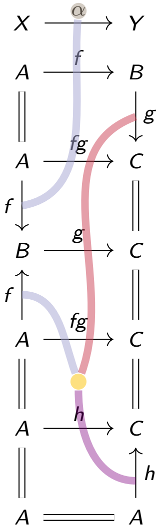
$$\dfrac{X \underset{f}{\Longrightarrow} Y}{X \underset{fg}{\Longrightarrow} g^+\, Y}\ \mathsf{R}g^+$$

$$\dfrac{X \underset{fg}{\Longrightarrow} g^+\, Y}{f^+\, X \underset{g}{\Longrightarrow} g^+\, Y}\ \mathsf{L}f^+$$

$$\dfrac{f^+\, X \underset{g}{\Longrightarrow} g^+\, Y}{f^-\, f^+\, X \underset{fg}{\Longrightarrow} g^+\, Y}\ \mathsf{L}f^-$$

$$\begin{array}{c} f^-\, f^+\, X \underset{fg}{\Longrightarrow} g^+\, Y \\ \hline\hline f^-\, f^+\, X \underset{h}{\Longrightarrow} g^+\, Y \end{array}$$

$$\dfrac{f^-\, f^+\, X \underset{h}{\Longrightarrow} g^+\, Y}{f^-\, f^+\, X \underset{\mathrm{id}_A}{\Longrightarrow} h^-\, g^+\, Y}\ \mathsf{R}h^-$$



Formulas

Judgment $S \sqsubset A$ ("S an object of $\mathcal{B}\mathrm{if}(p)$ lying over A")

Formation rules:

$$\frac{X \in \mathcal{D} \quad p(X) = A}{X \sqsubset A} \qquad \frac{S \sqsubset A \quad f : A \to B}{f^{+} S \sqsubset B} \qquad \frac{f : A \to B \quad T \sqsubset B}{f^{-} T \sqsubset A}$$

Derivations

Judgment $\alpha : S \Longrightarrow_h T$ ("$\alpha : S \to T$ represents an arrow of $\mathcal{B}\mathrm{if}(p)$ lying over h")

Inference rules:

$$\frac{\delta : X \to Y \in \mathcal{D} \qquad p(\delta) = f}{\delta : X \underset{f}{\Longrightarrow} Y}\ \delta$$

$$\frac{\alpha : T \underset{g'}{\Longrightarrow} T'}{\overline{g}.\alpha : g^{-}\ T \underset{gg'}{\Longrightarrow} T'}\ \mathrm{L}g^{-} \qquad\qquad \frac{\alpha : S' \underset{f'}{\Longrightarrow} S}{\alpha.f : S' \underset{f'f}{\Longrightarrow} f^{+}\ S}\ \mathrm{R}f^{+}$$

$$\frac{\alpha : S \underset{fg}{\Longrightarrow} T}{f \backslash_g\ \alpha : f^{+}\ S \underset{g}{\Longrightarrow} T}\ \mathrm{L}f^{+} \qquad\qquad \frac{\alpha : S \underset{fg}{\Longrightarrow} T}{\alpha\ {}_f/\ \overline{g} : S \underset{f}{\Longrightarrow} g^{-}\ T}\ \mathrm{R}g^{-}$$

Permutation equivalences (in proof-term syntax)

$$(\overline{f}.\alpha).h \sim \overline{f}.(\alpha.h) \qquad \text{for } \alpha \text{ over } g \tag{1}$$
$$(f \backslash_g \alpha).h \sim f \backslash_{gh} (\alpha.h) \qquad \text{for } \alpha \text{ over } fg \tag{2}$$
$$(\overline{f}.\alpha)\ {}_{fg}/\ \overline{h} \sim \overline{f}.(\alpha\ {}_{g}/\ \overline{h}) \qquad \text{for } \alpha \text{ over } gh \tag{3}$$
$$(f \backslash_{gh} \alpha)\ {}_{g}/\ \overline{h} \sim f \backslash_g (\alpha\ {}_{fg}/\ \overline{h}) \qquad \text{for } \alpha \text{ over } fgh \tag{4}$$

Identity and Composition

Identity derivations defined by induction on the formula:

$$\frac{S \sqsubset A}{S \underset{\mathrm{id}_A}{\Longrightarrow} S}\ \mathrm{id}_S \qquad \mathrm{id}_{f^+\,S} \stackrel{\text{def}}{=} f \backslash_{\mathrm{id}_B} (\mathrm{id}_S.f) \qquad \mathrm{id}_{g^-\,T} \stackrel{\text{def}}{=} (\overline{g}.\mathrm{id}_T)\ {}_{\mathrm{id}_B}/\ \overline{g}$$

Composition is cut-elimination:

$$\frac{\alpha : S \underset{g}{\Longrightarrow} T \qquad \beta : T \underset{h}{\Longrightarrow} U}{\alpha \cdot \beta : S \underset{gh}{\Longrightarrow} U}$$

Identity and Composition

Principal cuts:

$$\begin{aligned} (\alpha.f) \cdot (f \backslash_h \beta) &\stackrel{\text{def}}{=} \alpha \cdot \beta \\ (\alpha\ {}_g/\ \overline{f}) \cdot (\overline{f}.\beta) &\stackrel{\text{def}}{=} \alpha \cdot \beta \end{aligned}$$

Commutative cuts:

$$\begin{aligned} (\overline{f}.\alpha) \cdot \beta &\stackrel{\text{def}}{=} \overline{f}.(\alpha \cdot \beta) & \qquad \alpha \cdot (\beta.f) &\stackrel{\text{def}}{=} (\alpha \cdot \beta).f \\ (f \backslash_g \alpha) \cdot \beta &\stackrel{\text{def}}{=} f \backslash_{gh} (\alpha \cdot \beta) & \qquad \alpha \cdot \beta\ {}_h/\ \overline{f} &\stackrel{\text{def}}{=} (\alpha \cdot \beta)\ {}_{gh}/\ \overline{f} \end{aligned}$$

Note: ambiguous cases up to equivalence.

Putting it all together

Let $\mathcal{B}\mathrm{if}(p)$ be the category whose objects are bifibrational formulas and whose arrows are $(\sim)$-equivalence classes of derivations.

Let Λ_p be the functor $\mathcal{B}\mathrm{if}(p) \to \mathcal{C}$ sending $(S \sqsubset A)$ to A and $(\alpha : S \Longrightarrow_f T)$ to f.

Theorem. $\Lambda_p : \mathcal{B}\mathrm{if}(p) \to \mathcal{C}$ is the free bifibration on $p : \mathcal{D} \to \mathcal{C}$.

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Theorem. $\Lambda_p : \mathcal{B}\mathrm{if}(p) \to \mathcal{C}$ is the free bifibration on $p : \mathcal{D} \to \mathcal{C}$.

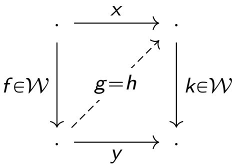
Note the construction easily and usefully generalizes to $(\mathcal{P}, \mathcal{N})$-fibrations, where one restricts the pushforward and pullback operations to some classes of arrows:

$$\frac{S \sqsubset A \qquad f : A \to B \in \mathcal{P}}{f^+ S \sqsubset B} \qquad\qquad \frac{g : B \to C \in \mathcal{N} \qquad T \sqsubset C}{g^- T \sqsubset B}$$

Canonicity

In general, permutation equivalence is undecidable! But using ideas from proof theory, we can derive unique normal forms for $\mathcal{B}\mathrm{if}(p)$ when $\mathcal{C}$ is *factorization preordered* (FP), or more generally for $\mathcal{B}\mathrm{if}(p, \mathcal{P}, \mathcal{N})$ when $\mathcal{C}$ is both $\mathcal{P}$-FP and $\mathcal{N}$-FP.

A category is $\mathcal{W}$-**factorization preordered** just in case every commuting square of the following form has at most one diagonal filler:



Canonicity gives us both decidability and enumeration!

5. Three examples of free bifibrations

Example #1

Let $p_2 : 1 \to 2$ be the functor shown below:

$$\begin{array}{ccc} 1 & & * & & \\ {\scriptstyle p_2}\big\downarrow & & & & \\ 2 & & 0 & \xrightarrow{f} & 1 \end{array}$$

Question: what is the fiber of $\mathcal{B}\mathrm{if}(p_2)$ over 0?

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$$
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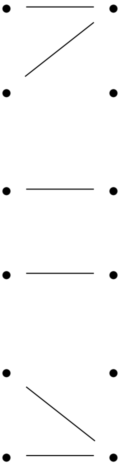
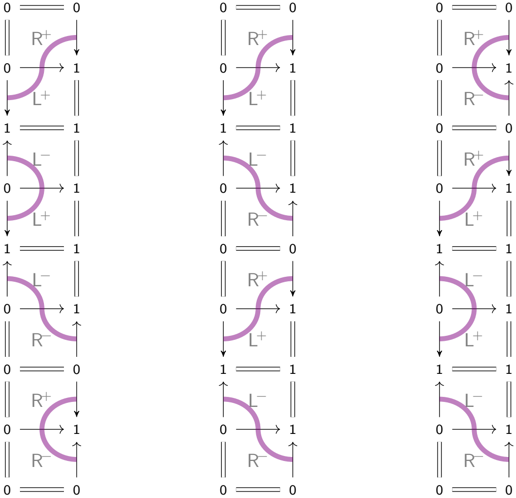
\end{array}
$$

Question: what is the fiber of $\mathcal{B}\mathrm{if}(p_2)$ over 0?

Answer: $\mathcal{B}\mathrm{if}(p_2)_0 \simeq \Delta$, the category of finite ordinals and order-preserving functions!




0 0
R+
0 1
L+
1 1
L−
0 1
L+
1 1
L−
0 1
R−
0 0
R+
0 1
R−
0 0
0 0
R+
0 1
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1 1
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R−
0 0
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0 1
L+
1 1
L−
0 1
R−
0 0
0 0
R+
0 1
R−
0 0
R+
0 1
L+
1 1
L−
0 1
L+
1 1
L−
0 1
R−
0 0

Example #2

Let ω be the totally ordered set of natural numbers considered as a category.

$$
\begin{array}{ccc}
1 & \qquad & * \\
\Big\downarrow{\scriptstyle p_\omega} & & \\
\omega & & 0 \xrightarrow{f_0} 1 \xrightarrow{f_1} 2 \xrightarrow{f_2} \cdots
\end{array}
$$

Question: what is the fiber of $\mathcal{B}\mathrm{if}(p_\omega)$ over 0?

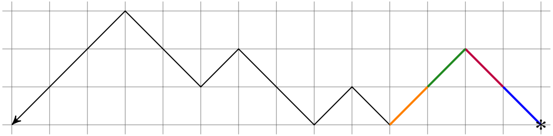
Example #2

Let ω be the totally ordered set of natural numbers considered as a category.

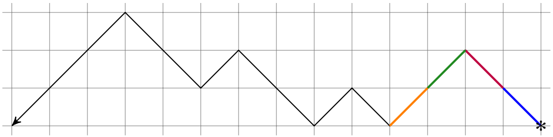
$$
\begin{array}{ccc}
1 & & * \\
\Big\downarrow p_\omega & & \\
\omega & & 0 \xrightarrow{f_0} 1 \xrightarrow{f_1} 2 \xrightarrow{f_2} \cdots
\end{array}
$$

Question: what is the fiber of $\mathcal{B}\mathrm{if}(p_\omega)$ over 0?

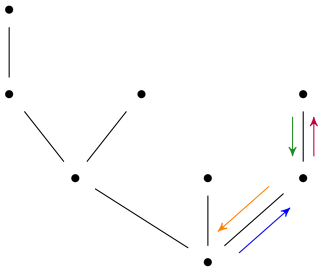
Answer: $\mathcal{B}\mathrm{if}(p_2)_0 \simeq \mathsf{PTree}$, the subcategory of $[\omega^{\mathrm{op}}, \Delta]$ spanned by finite plane trees!

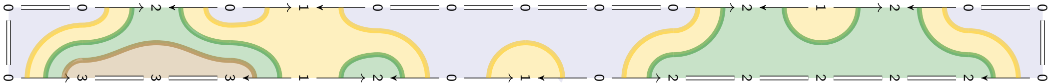


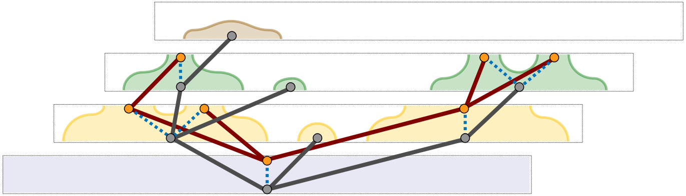
$f^- \; f^- \; f^- \; f^+ \; f^+ \; f^- \; f^+ \; f^+ \; f^- \; f^+ \; f^- \; f^- \; f^+ \; f^+ \; *$

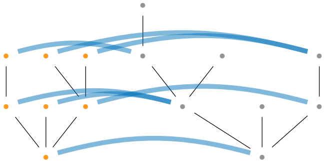


$f^- \ f^- \ f^- \ f^+ \ f^+ \ f^- \ f^+ \ f^+ \ f^- \ f^+ \ f^- \ f^- \ f^+ \ f^+ \ *$









Example #3

Let $i : \mathbb{N} \to \Delta$ be the inclusion of the natural numbers (seen as a discrete category) into the simplex category, and build the free **(epi,mono)-fibration** generated by i.

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Next quotient by the Beck-Chevalley condition for bicartesian squares:

$$\begin{array}{ccc} \cdot & \overset{c}{\twoheadrightarrow} & \cdot \\ a \downarrow & \mathrm{BC} & \downarrow d \\ \cdot & \underset{b}{\twoheadrightarrow} & \cdot \end{array} \qquad c^{+}\, a^{-} \overset{\sim}{\Longrightarrow} d^{-}\, b^{+}$$

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Question: what does this give you?

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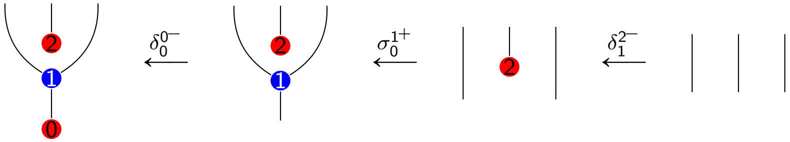
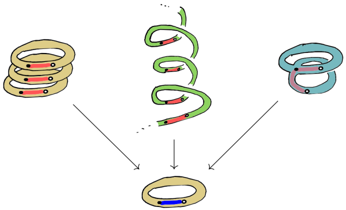
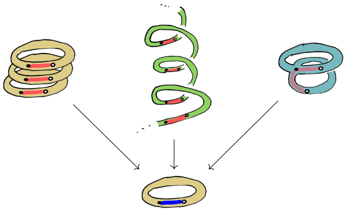
Question: what does this give you? Answer: the lattice of noncrossing partitions!

$\delta_0^{0-}\ \sigma_0^{1+}\ \delta_1^{2-}\ 3$:

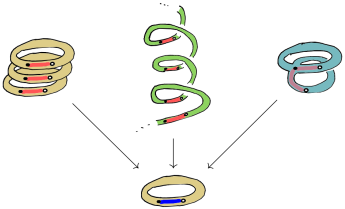


Conclusion





A variety of computational and deductive systems may be considered as bundles of categories, in a way that allows to treat them uniformly.



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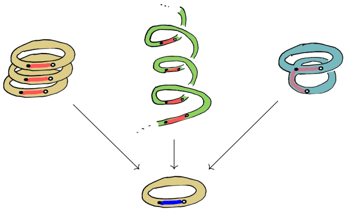
Fibrational properties as well as free constructions play an important role, and have deep connections to logic.



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References:

- w/P.-A. Melliès: "Functors are type refinement systems" (POPL 2015); "The categorical contours of the Ch.-Sch. repr. theorem" (LMCS 21:2, 2025)
- w/B. Clarke & G. Scherer: "The free bifibration on a functor", arxiv:2511.07314