Polarity and the Logic of Delimited Continuations

Noam Zeilberger

Université Paris 7

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Part I

Questions

Some old questions and old answers

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Derived by $\neg\neg$ translations into intuitionistic logic.

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Q: What are the meanings of programs with effects?

Some old questions and old answers

Q: What are the meanings of proofs in classical logic?
A [Kolmogorov, Glivenko-Kuroda, Gödel-Gentzen, . . .]:
Derived by $\neg\neg$ translations into intuitionistic logic.

Q: What are the meanings of programs with effects?
A [Reynolds, Steele-Sussman, Plotkin, ...]:
Derived by CPS translations into lambda calculus.

Refining these answers. . .

Of course As beg more Qs: meaning of proofs/programs in IL/λC?

Refining these answers. . .

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(Yes, answers are well-known, but we can also dodge the question!. . .)

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Idea: directly study **canonical forms** in *image* of translations, e.g., as. . .

- strategies (game semantics)
- focusing proofs (proof theory)

Polarity is a guide for describing these canonical forms
Internalized as **polarized logic**

. . . leading to another question. . .

Polarity ($\neg\neg$-translation, CPS) plays a role in constructivizing classical logic. Does it have a role in *constructive logic*?[1]

[1] Cf. Intuitionistic focusing, Benton's LNL logic, Watkins' CLF, Levy's CBPV, ...

. . . and another

Delimited continuations greatly widen the scope of continuation semantics.[2] What is their logical structure?

[2] Cf. Felleisen '88, Danvy & Filinski '90, Filinski '94, Shan Ph.D., . . .

Towards positive answers

Key (simple) idea: study polarity with more than one **answer type**

- Introduces asymmetry between positive and negative polarity
- Yields different "$\neg\neg$"-interpretations of intuitionistic logic
- *Positive* answer types give rise to monadic effects

Paper works out this idea guided mainly by proof-theoretic principles

- pros: concrete, close connection between syntax and semantics
- cons: perhaps not so transparent, very partial picture

Part II

Review of Classical Polarity

The basic type distinction

P		N
defined by truth	vs.	defined by falsehood
(i.e., datatypes)		(e.g., records, classes, etc.)

The basic judgments

	Interpretation	
	Logical	Operational
$[P]$	"P obvious"	value of type P
$\bullet P$	"P false"	continuation accepting P
N	"N true"	value of type N
$[\bullet N]$	"N absurd"	continuation accepting N
$\#$	"contradiction"	well-typed expression

Syntax/Semantics

How to explain the meanings of the judgments? Different approaches…

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- definition-by-canonical-forms [most precise, primary in paper]
- definition-by-translation [shortly. . .]
- definition-by-handwaving [now!]

Intuition

$$\frac{\text{“direct proof of } P\text{”}}{[P]}$$

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$$\frac{\text{"direct proof of } P\text{"}}{[P]} \qquad \frac{\text{"direct proof of } P\text{"} \quad \longrightarrow \quad \#}{\bullet P}$$

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$$\frac{\text{“direct refutation of } N\text{”}}{[\bullet N]}$$

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$$\frac{[P] \quad \bullet P}{\#} \qquad \frac{N \quad [\bullet N]}{\#}$$

Definition by translation

Target: fragment of intuitionistic logic (or intuitionistic linear logic)

Given translations P^{+} and N^{-} [next slide], translate judgments J^{*} by:

Definition by translation

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Given translations P^+ and N^- [next slide], translate judgments J^* by:

$$[P]^* = P^+$$

Definition by translation

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Given translations P^+ and N^- [next slide], translate judgments J^* by:

$$[P]^* = P^+ \qquad \bullet P^* = P^+ \supset \#$$

(where $\#$ a distinguished logical atom)

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Given translations P^{+} and N^{-} [next slide], translate judgments J^{*} by:

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Given translations P^+ and N^- [next slide], translate judgments J^* by:

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Given translations P^+ and N^- [next slide], translate judgments J^* by:

$$[P]^* = P^+ \qquad \bullet P^* = P^+ \supset \#$$

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$$\#^* = \#$$

(where $\#$ a distinguished logical atom)

Definition by translation

Some connectives:

$$1^+ = \mathrm{T} = \bot^- \qquad 0^+ = \mathrm{F} = \top^-$$

$$(P_1 \otimes P_2)^+ = P_1^+ \wedge P_2^+ \qquad (N_1 \invamp N_2)^- = N_1^- \wedge N_2^-$$

$$(P_1 \oplus P_2)^+ = P_1^+ \vee P_2^+ \qquad (N_1 \& N_2)^- = N_1^- \vee N_2^-$$

$$(N^\perp)^+ = N^- \qquad (P \to N)^- = P^+ \wedge N^-$$

$$(\downarrow N)^+ = N^- \supset \# \qquad (\uparrow P)^- = P^+ \supset \#$$

The classical connection

Define "polarity-collapsing" translation:

$$|\otimes| = |\&| = \wedge \quad |\oplus| = |\parr| = \vee \quad |\rightarrow| = \supset \quad |-^{\perp}| = \neg \quad |{\downarrow}| = |{\uparrow}| = \cdot$$

Proposition

$\vdash^c |N|$ *iff* $\vdash^i N^*$ $\quad$ $\vdash^c \neg|P|$ *iff* $\vdash^i (\bullet P)^*$

Punchline: different polarizations yield different ¬¬-translations

Definition by canonical forms

Contexts $\Delta, \Gamma \ ::= \ \cdot \mid \Delta_1, \Delta_2 \mid N \mid \bullet P$

$$\frac{\Delta \Vdash [P] \quad \Gamma \vdash \Delta}{\Gamma \vdash [P]} \qquad \frac{\Delta \Vdash [P] \quad \longrightarrow \quad \Gamma, \Delta \vdash \#}{\Gamma \vdash \bullet P}$$

$$\frac{\Delta \Vdash [\bullet N] \quad \longrightarrow \quad \Gamma, \Delta \vdash \#}{\Gamma \vdash N} \qquad \frac{\Delta \Vdash [\bullet N] \quad \Gamma \vdash \Delta}{\Gamma \vdash [\bullet N]}$$

$$\frac{N \in \Gamma \quad \Gamma \vdash [\bullet N]}{\Gamma \vdash \#} \qquad \frac{\bullet P \in \Gamma \quad \Gamma \vdash [P]}{\Gamma \vdash \#} \qquad \frac{}{\Gamma \vdash \cdot} \qquad \frac{\Gamma \vdash \Delta_1 \quad \Gamma \vdash \Delta_2}{\Gamma \vdash \Delta_1, \Delta_2}$$

Definition by canonical forms

$$\frac{p \quad \sigma}{V^+}\ p[\sigma] \qquad \frac{p \quad \mapsto \quad E_p}{K^+}\ p{\mapsto}E_p$$

$$\frac{d \quad \mapsto \quad E_d}{V^-}\ d{\mapsto}E_d \qquad \frac{d \quad \sigma}{K^-}\ d[\sigma]$$

$$\frac{v \quad K^-}{E}\ v\ K^- \qquad \frac{k \quad V^+}{E}\ k\ V^+ \qquad \overline{\sigma}\ \cdot \qquad \frac{\sigma_1 \quad \sigma_2}{\sigma}\ (\sigma_1,\sigma_2)$$

Part III

Towards Generalized Polarity

Symmetry

Elegant symmetry or silly redundancy?

$[P]$	$\bullet P$
N	$[\bullet N]$

Symmetry

Elegant symmetry or silly redundancy?

$$\begin{array}{c|c} [P] & \bullet P \\ \hline N & [\bullet N] \end{array}$$

$$\Updownarrow$$

$$[P] \mid \bullet P \qquad \frac{[P]}{N} \qquad \frac{\bullet P}{[\bullet N]} \qquad N \mid [\bullet N]$$

Symmetry

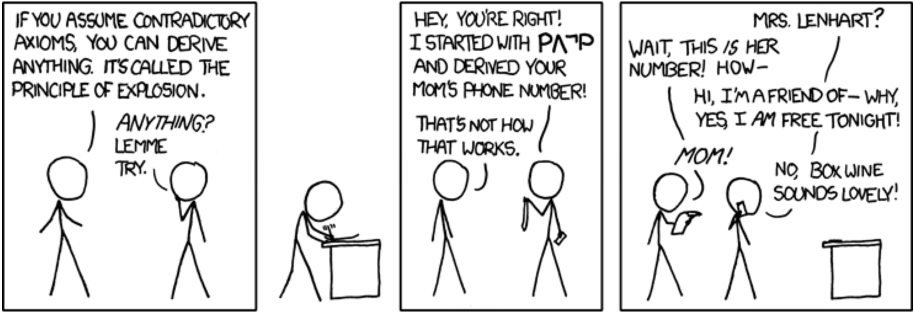
Elegant symmetry or silly redundancy?

$$\begin{array}{c|c} [P] & \bullet P \\ \hline N & [\bullet N] \end{array}$$

$$\Updownarrow$$

$$[P] \mid \bullet P \qquad \frac{[P]}{N} \qquad \frac{\bullet P}{[\bullet N]} \qquad N \mid [\bullet N]$$

Before giving up on our intuitions, let's think about "contradiction" $\#$...



Credit: Randall Munroe

From Symmetry to Asymmetry

Key (simple) idea:

$\boxed{\# \rightsquigarrow P}$ $\boxed{\bullet A \rightsquigarrow A \triangleright P}$

Perfect symmetry between positive and negative broken!

The basic judgments

	Interpretation	
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The basic judgments++

	Interpretation	
	Logical	Operational
$[P]$	"P obvious"	value of type P
$P_1 \triangleright P_2$	"P_1 entails P_2"	continuation from P_1 to P_2
N	"N true"	value of type N
$[\bullet N]$	"N absurd"	continuation accepting N
P	"P true"	expression of type P

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	Interpretation	
	Logical	Operational
$[P]$	"P obvious"	value of type P
$P_1 \triangleright P_2$	"P_1 entails P_2"	continuation from P_1 to P_2
N	"N true"	value of type N
$[N \triangleright P]$	"N manifests P"	continuation from N to P
P	"P true"	expression of type P

Intuition

$$\frac{\text{"direct proof of } P\text{"}}{[P]} \qquad \frac{\text{"direct proof of } P\text{"} \quad \longrightarrow \quad \#}{\bullet P}$$

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$$\frac{[P] \quad \bullet P}{\#} \qquad \frac{N \quad [\bullet N]}{\#}$$

Intuition++

$$\frac{\text{“direct proof of } P\text{”}}{[P]} \qquad \frac{\text{“direct proof of } P_1\text{”} \quad \longrightarrow \quad P_2}{P_1 \triangleright P_2}$$

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$$\frac{[P] \quad \bullet P}{\#} \qquad \frac{N \quad [\bullet N]}{\#}$$

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$$\frac{\text{“direct argument from } N \text{ to } \alpha\text{”} \quad \longrightarrow \quad \alpha}{N} \qquad \frac{\text{“direct argument from } N \text{ to } P\text{”}}{[N \triangleright P]}$$

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$$\frac{[P] \quad P \triangleright P'}{P'} \qquad \frac{N \quad [N \triangleright P]}{P}$$

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$$\frac{[P] \quad P \triangleright P'}{P'} \qquad \frac{N \quad [N \triangleright P]}{P}$$

$$\frac{[P]}{P} \qquad \frac{P \quad P \triangleright P'}{P'}$$

Definition by translation

Target: fragment of intuitionistic logic (or intuitionistic linear logic)

Given type translations P^+ and N^-, translate judgments by:

$$[P]^* = P^+ \qquad \bullet P^* = P^+ \supset \#$$

$$N^* = N^- \supset \# \qquad [\bullet N]^* = N^-$$

$$\#^* = \#$$

(where $\#$ a distinguished logical atom)

Definition by translation++

Target: fragment of 2nd-order intuitionistic logic

Given type translations P^+ and $N^{-\alpha}$, translate judgments by:

$$[P]^* = P^+ \qquad (P_1 \triangleright P_2)^* = {P_1}^+ \supset \quad {P_2}^+$$

$$N^* = \forall\alpha.N^{-\alpha} \supset \quad \alpha \qquad [N \triangleright P]^* = N^{-\alpha}[P^+/\alpha]$$

$$P^* = \quad P^+$$

Definition by translation++

Target: fragment of 2nd-order intuitionistic logic + “monad T”

Given type translations P^+ and $N^{-\alpha}$, translate judgments by:

$$[P]^* = P^+ \qquad (P_1 \triangleright P_2)^* = {P_1}^+ \supset T{P_2}^+$$
$$N^* = \forall\alpha.N^{-\alpha} \supset T\alpha \qquad [N \triangleright P]^* = N^{-\alpha}[P^+/\alpha]$$
$$P^* = TP^+$$

(where “monad T” = $[\forall\alpha.\alpha \supset T\alpha] \wedge [\forall\alpha\beta.(\alpha \supset T\beta) \supset (T\alpha \supset T\beta)]$)

Definition by translation++

Type translation:

$$1^{+} = \mathrm{T} = \bot^{\text{-}\alpha} \qquad 0^{+} = \mathrm{F} = \top^{\text{-}\alpha}$$

$$(P_1 \otimes P_2)^{+} = P_1^{+} \wedge P_2^{+} \qquad (N_1 ⅋ N_2)^{\text{-}\alpha} = N_1^{\text{-}\alpha} \wedge N_2^{\text{-}\alpha}$$

$$(P_1 \oplus P_2)^{+} = P_1^{+} \vee P_2^{+} \qquad (N_1 \& N_2)^{\text{-}\alpha} = N_1^{\text{-}\alpha} \vee N_2^{\text{-}\alpha}$$

$$(N \multimap\!\!\bullet\, P)^{+} = N^{\text{-}\alpha}[P^{+}/\alpha] \qquad (P \to N)^{\text{-}\alpha} = P^{+} \wedge N^{\text{-}\alpha}$$

$$(\downarrow N)^{+} = \forall \alpha . N^{\text{-}\alpha} \supset T\alpha \qquad (\uparrow P)^{\text{-}\alpha} = P^{+} \supset T\alpha$$

The intuitionistic connection

Define "polarity-collapsing" translation:

$$|\otimes| = |\&| = \wedge \quad |\oplus| = |\invamp| = \vee \quad |{\to}| = |{\multimap\!\!\bullet}| = \supset \quad |{\downarrow}| = |{\uparrow}| = \cdot$$

Proposition

$\vdash^i |A|$ *iff* $Mon_T \vdash^{2i} A^*$ *for* $\invamp$*,* $\multimap\!\!\bullet$*-free* A

Punchline: different "¬¬"-interpretations **of** intuitionistic logic

"Polarized IL is a restriction of a generalization of polarized CL"

Definition by canonical forms++

Contexts $\Delta, \Gamma \;::=\; \cdot \mid \Delta_1, \Delta_2 \mid N \mid P \triangleright P'$

$$\frac{\Delta \Vdash [P] \quad \Gamma \vdash \Delta}{\Gamma \vdash [P]} \qquad \frac{\Delta \Vdash [P] \quad \longrightarrow \quad \Gamma, \Delta \vdash P'}{\Gamma \vdash P \triangleright P'}$$

$$\frac{\alpha.\Delta \Vdash [N] \triangleright - \quad \longrightarrow \quad \Gamma, \alpha.\Delta \vdash \alpha}{\Gamma \vdash N} \qquad \frac{\alpha.\Delta \Vdash [N] \triangleright - \quad \Gamma \vdash \Delta[P/\alpha]}{\Gamma \vdash [N] \triangleright P}$$

$$\frac{N \in \Gamma \quad \Gamma \vdash [N] \triangleright P}{\Gamma \vdash P} \qquad \frac{P \triangleright P' \in \Gamma \quad \Gamma \vdash [P]}{\Gamma \vdash .P'} \qquad \frac{}{\Gamma \vdash \cdot} \qquad \frac{\Gamma \vdash \Delta_1 \quad \Gamma \vdash \Delta_2}{\Gamma \vdash \Delta_1, \Delta_2}$$

$$\frac{\Gamma \vdash [P]}{\Gamma \vdash P} \qquad \frac{\Gamma \vdash .P \quad \Gamma \vdash P \triangleright P'}{\Gamma \vdash P'}$$

Definition by canonical forms++

$$\frac{p \quad \sigma}{V^{+}}\; p[\sigma] \qquad\qquad \frac{p \quad \mapsto \quad E_p}{K^{+}}\; p{\mapsto}E_p$$

$$\frac{d \quad \mapsto \quad E_d}{V^{-}}\; d{\mapsto}E_d \qquad\qquad \frac{d \quad \sigma}{K^{-}}\; d[\sigma]$$

$$\frac{v \quad K^{-}}{E}\; v\ K^{-} \qquad \frac{k \quad V^{+}}{.E}\; k\ V^{+} \qquad \frac{}{\sigma}\; \cdot \qquad \frac{\sigma_1 \quad \sigma_2}{\sigma}\; (\sigma_1,\sigma_2)$$

$$\frac{V^{+}}{E}\; !V^{+} \qquad\qquad \frac{.E \quad K^{+}}{E}\; K^{+}\$.E$$

The delimited connection

Delimited control operators are already here, really!

- Danvy & Filinski's original type-and-effect system as derived rules
- Connections to Asai & Kameyama '07 and Kiselyov & Shan '07
- See paper (and Twelf code!) for a more concrete connection

Important caveat: **only the first-level of the CPS hierarchy**

Inconclusions

Asymmetry in constructive logic is still not very well-understood

Continuation semantics ($\neq$ semantics of callcc) deserves to be revisited

Filinski's **monadic reflection** [POPL94/10] is an underappreciated idea

The **CPS hierarchy** (= "substructural hierarchy"?) is ripe for exploration