# Parsing as a lifting problem and the Chomsky-Schützenberger Representation Theorem

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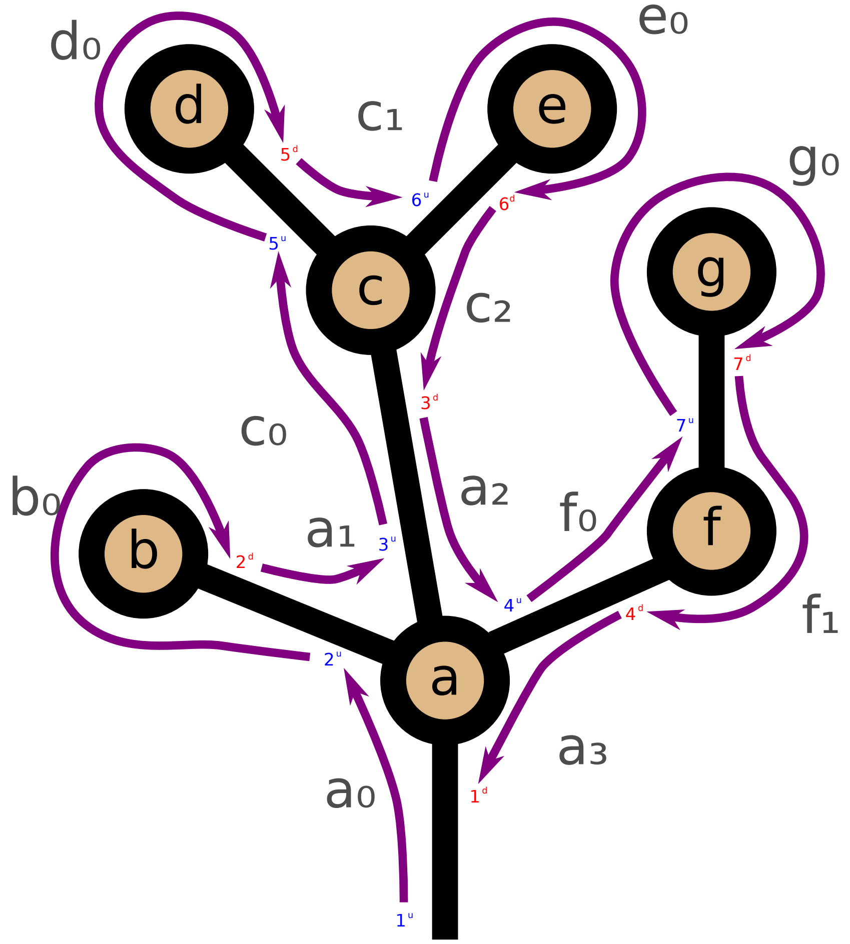



based on a paper presented at MFPS 2022

preliminary version: https://hal.archives-ouvertes.fr/hal-03702762 (comments welcome!)

# 1. Introduction

# A functorial view of type systems

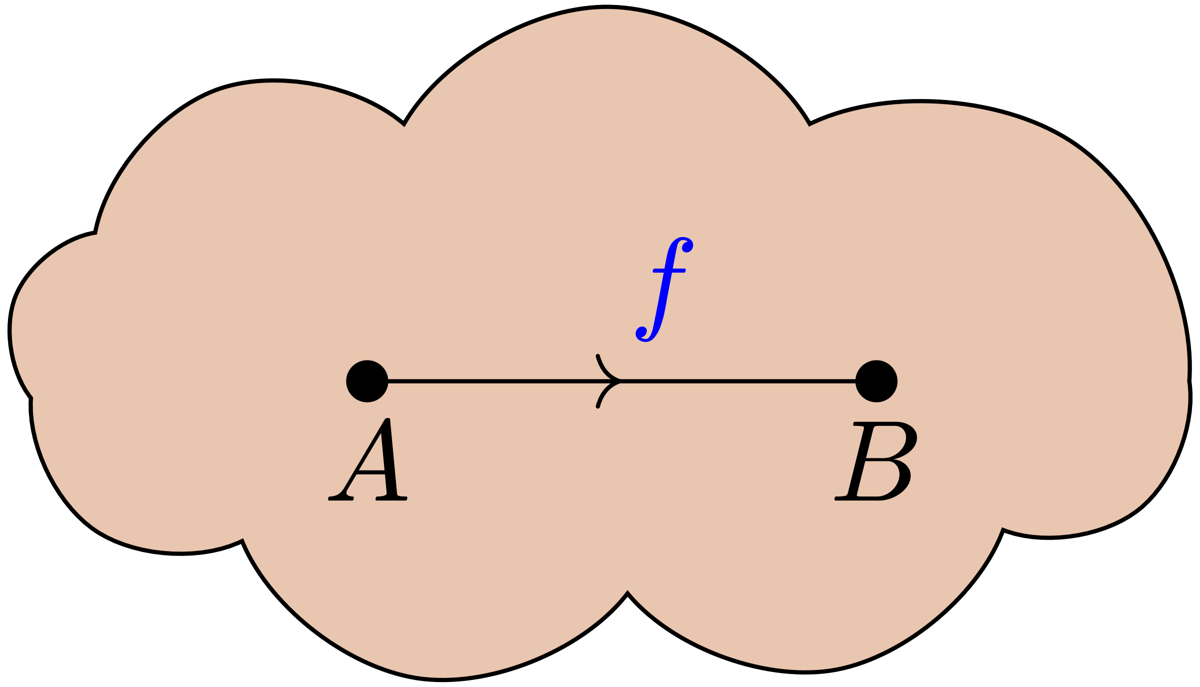
(cf. M&Z, "Functors are Type Refinement Systems", POPL 2015)

## Manifesto.

The standard interpretation of type systems as categories *collapses the distinction* between terms, typing judgments, and typing derivations, and is *therefore inadequate* for understanding what type systems do mathematically. Instead, type systems are better modelled as **functors** $p : \mathbb{D} \to \mathbb{T}$ from a category $\mathbb{D}$ whose morphisms are typing derivations to a category $\mathbb{T}$ whose morphisms are the terms corresponding to the *underlying subjects of those derivations.*

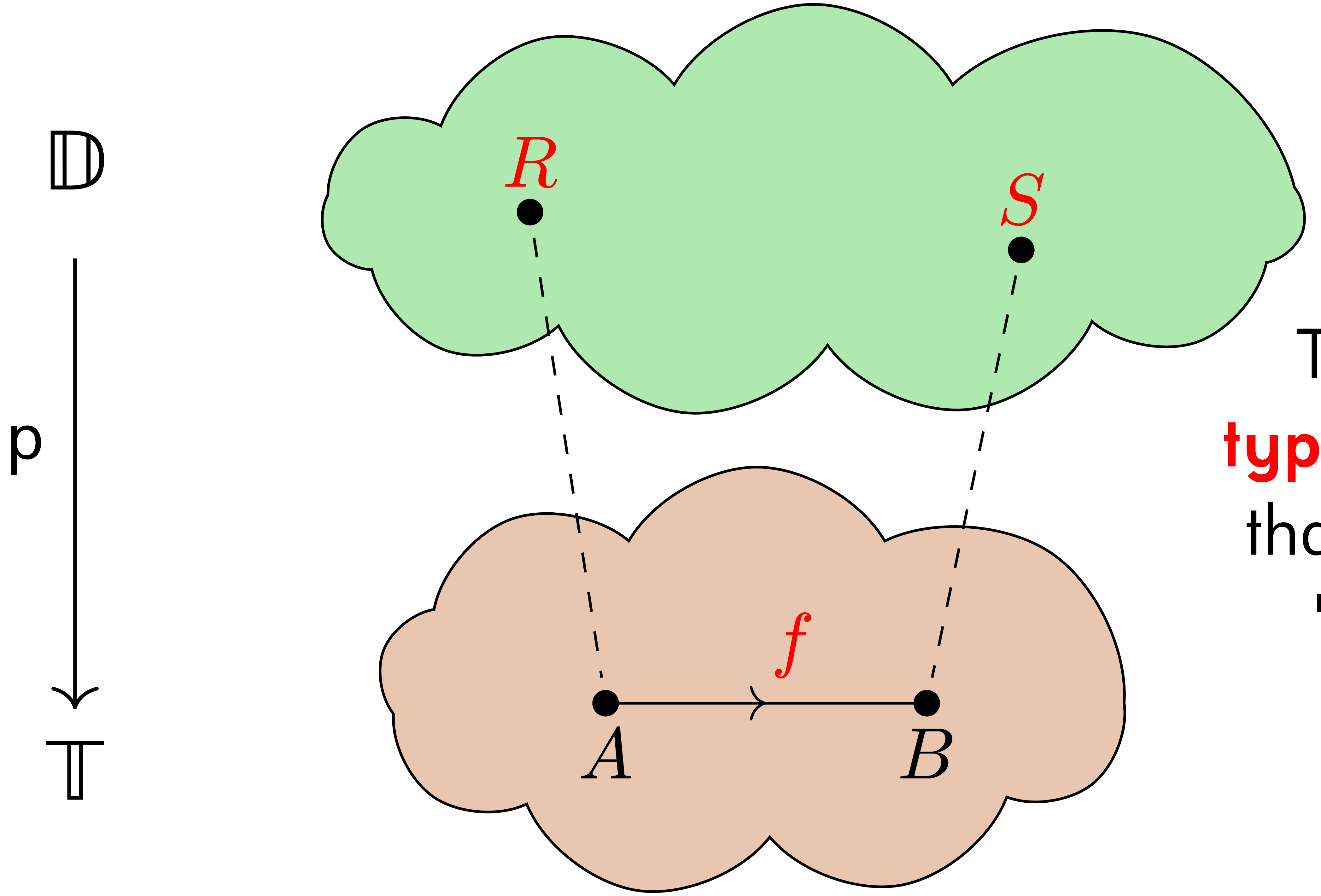
# Typing as a lifting problem

$\mathbb{T}$



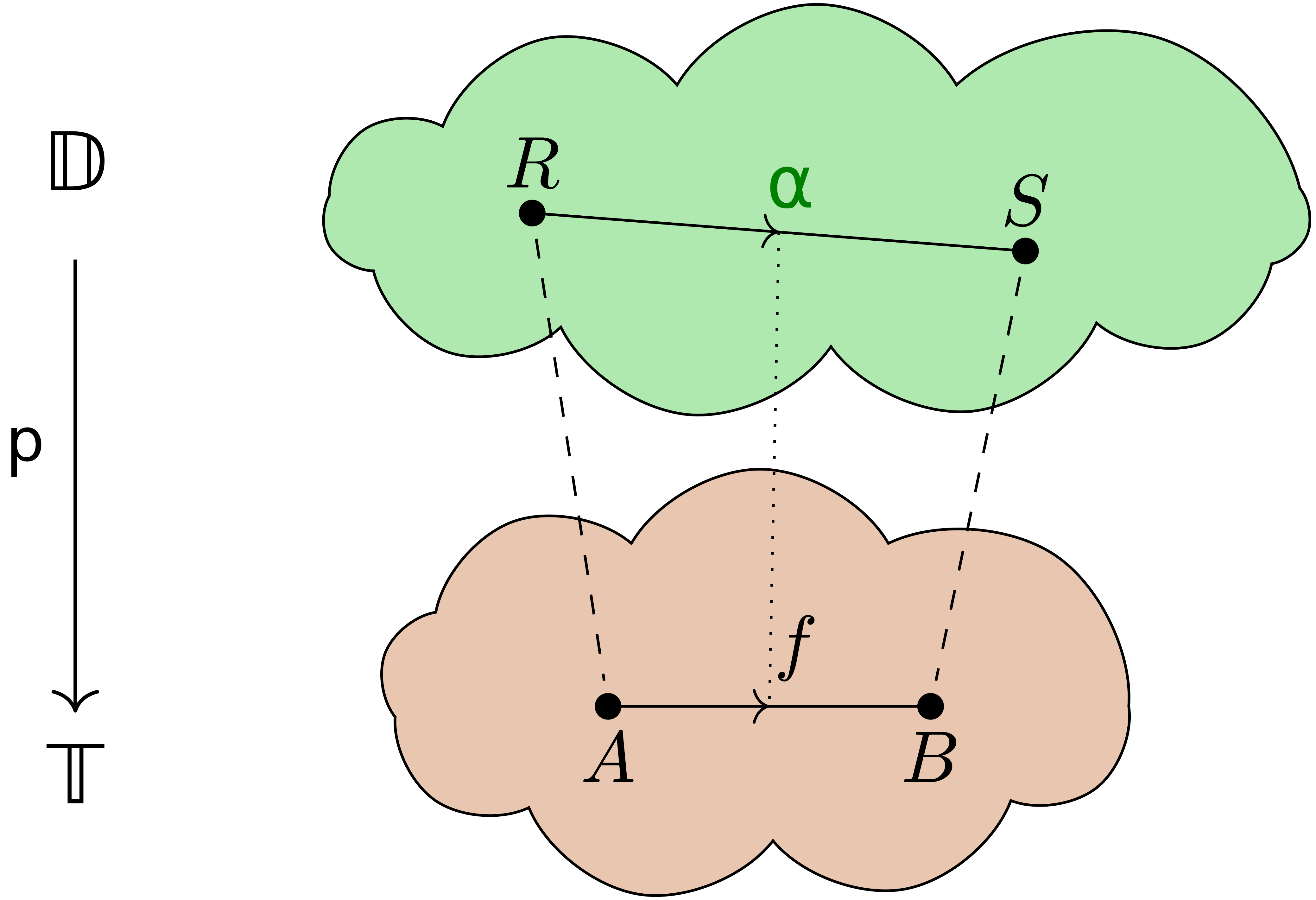

f is a **term** with
"intrinsic" type A → B

# Typing as a lifting problem




The triple (R,f,S) form a **typing judgment**, asserting that f may be assigned an "extrinsic" type R → S

# Typing as a lifting problem




α is a **typing derivation** providing evidence for the judgment

# A functorial view of context-free grammars

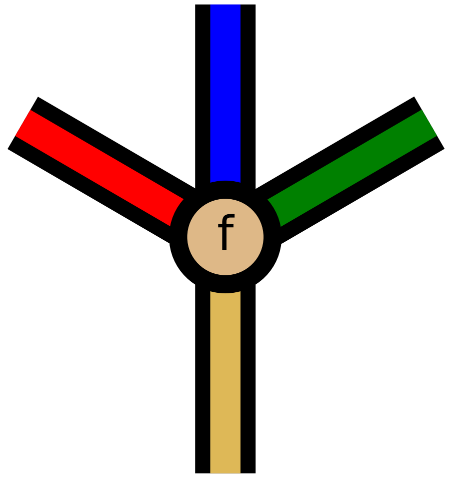
We developed this perspective in a series of papers, and believe it may be usefully applied to a large variety of deductive systems, beyond type systems in the traditional sense. In this work, we focus on derivability in context-free grammars, a classic topic in formal language theory with wide applications in CS.

Our starting point will be to *represent CFGs as **functors of operads*** $p : \mathbb{D} \to \mathbb{T}$, where $\mathbb{D}$ is a freely generated (colored) operad and $\mathbb{T} = W[\Sigma]$ is something we call the "operad of spliced words".

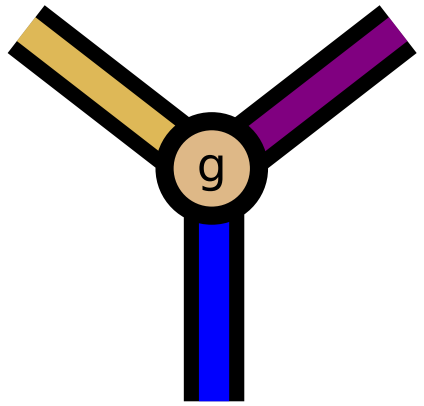
# Reminder on operads

(Usage note: "operad" = colored operad = multicategory.)

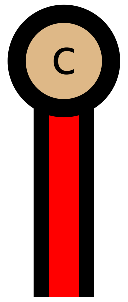
***operations***




$f : R,B,G \to Y$

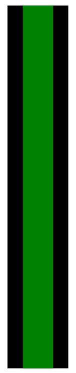



$g : Y,P \to B$



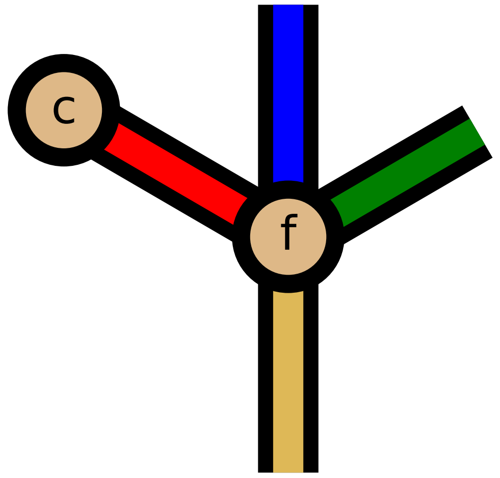

$c : R$

***identity***

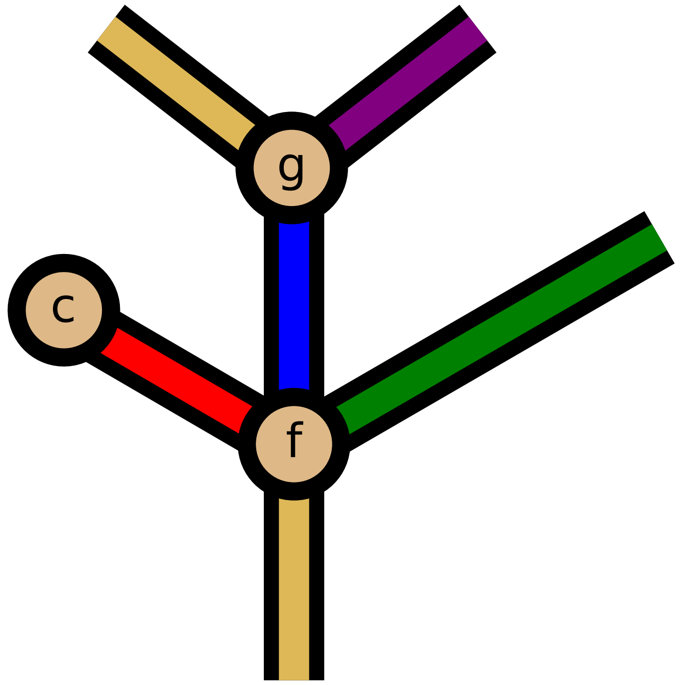


$id_G : G \to G$

***partial / parallel composition***




$f \circ_0 c : B,G \to Y$




$f \circ (c,g,id_G) : Y,P,G \to Y$

***+ associativity & unitality axioms***

# Reminder on CFGs

A context-free grammar is a tuple $G = (\Sigma, N, S, P)$ consisting of:

- a finite set $\Sigma$ of *terminal symbols*
- a finite set $N$ of *non-terminal symbols*
- a distinguished element $S \in N$ called the *start symbol*
- a finite set $P$ of *production rules* $R \to \sigma$ where $R \in N$ and $\sigma \in (N \cup \Sigma)^*$

We write $\sigma_1 \Rightarrow \sigma_2$ if there exist $\rho, \tau \in (N \cup \Sigma)^*$ and a production rule $R \to \sigma$ such that $\sigma_1 = \rho R \tau$, $\sigma_2 = \rho\sigma\tau$. The *language* of G is the set of strings $\{ w \in \Sigma^* \mid S \Rightarrow^+ w \}$.

# The operad of spliced words

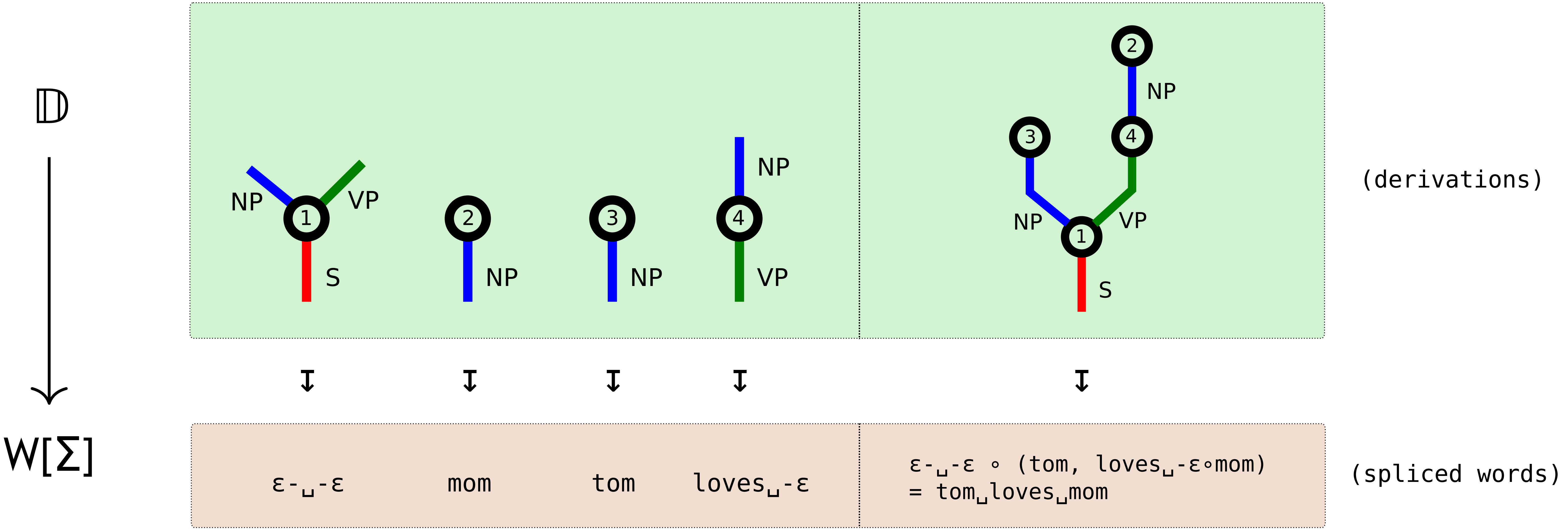
Observation: any production rule can be factored as
$R \to w_0 R_1 w_1 \ldots R_n w_n$, where $w_0, w_1, \ldots, w_n \in \Sigma^*$ and $R_1, \ldots, R_n \in N$.

If we forget the non-terminals, the remaining sequence $w_0\text{-}w_1\text{-}\ldots\text{-}w_n$ can be seen as an n-ary operation of the ***operad of spliced words*** $W[\Sigma]$. Composition in this (uncolored) operad is performed by "splicing into the gaps", for example:

$$(\text{ha-ha-ha}) \circ (\text{foo}, \text{bar-baz}) = \text{hafoohabar-bazha}$$

# Representing CFGs as functors of operads: example

```
1 : S → NP VP
2 : NP → mom
3 : NP → tom
4 : VP → loves NP
```



# Plan for the talk

It turns out that taking "spliced words" extends to a functor W[-] : Cat → Operad, allowing us to define CFGs of arrows over any category. We'll see that representing CFGs as functors leads to a simplification of many standard concepts, and that closure properties of CF languages generalize to CF languages of arrows.

Later, we will see that W[-] has a left adjoint C[-] : Operad → Cat. This consruction, called the "contour category" of an operad, has a nice geometric interpretation, and we will use it to prove (a refinement and generalization of) the Chomsky-Schützenberger Representation Theorem*.

In between, we will also talk about automata over categories and operads.

*original version: « any CF language is the homomorphic image of the intersection of a Dyck language with a regular language »

# Related work

The idea of defining CFGs as functors from free multicategories was discussed briefly by R.F.C. Walters in "A note on context-free languages", JPAA 62 (1989)

This idea is also closely related to Philippe de Groote's encoding of CFGs as *abstract categorial grammars,* although the ACG work is expressed within a λ-calculus framework rather than a categorical / operadic one.

See introduction to our paper for a bit more discussion of related work.
Additional pointers to related work are of course welcome.
(Has the contour / splicing adjunction not been noticed before??)

# 2. Context-free languages of arrows

# The operad of spliced arrows

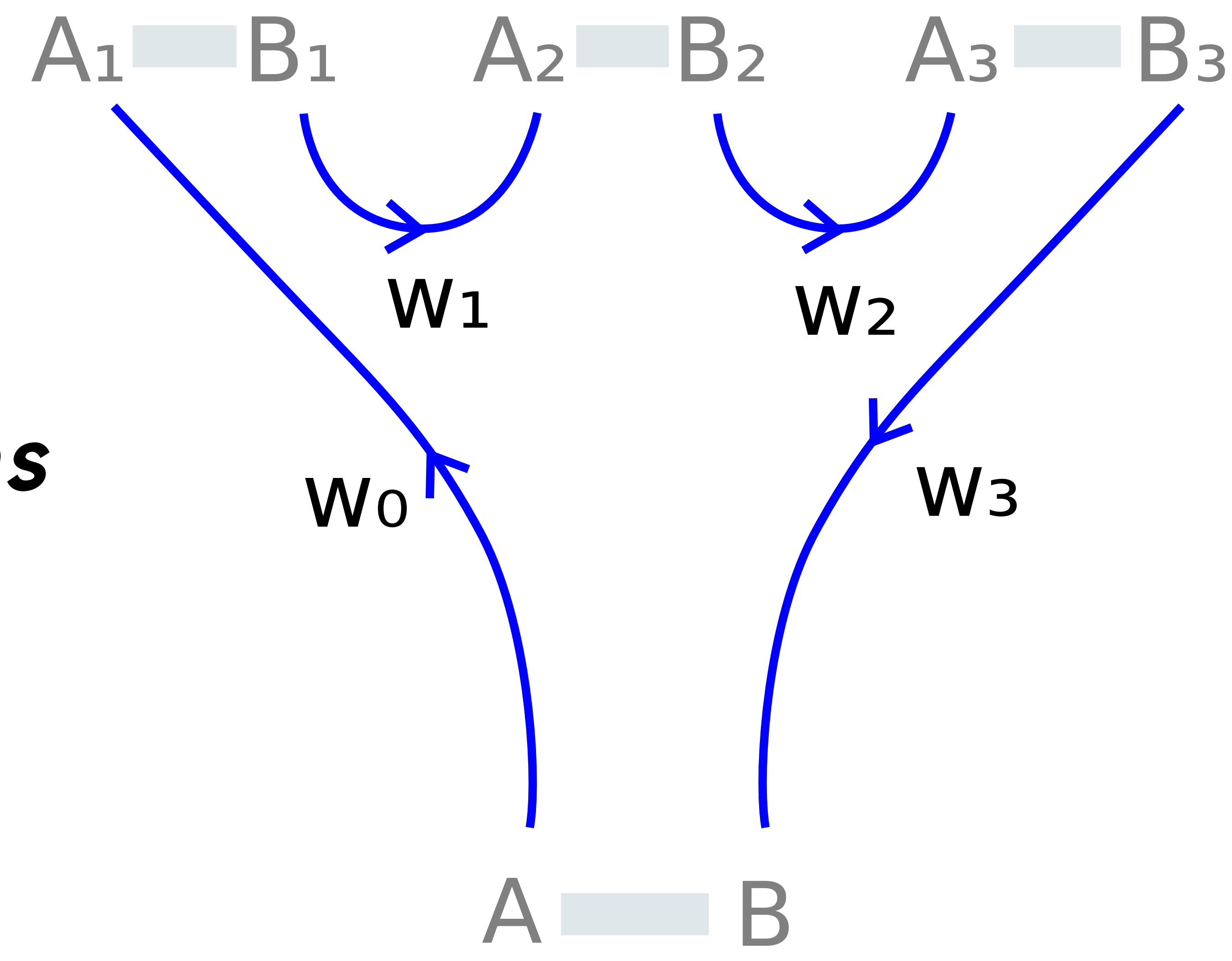
Let $\mathbb{C}$ be a category. The operad W[$\mathbb{C}$] is defined as follows:

- its colors are pairs (A,B) of objects of $\mathbb{C}$;
- its n-ary operations $(A_1,B_1),\cdots,(A_n,B_n) \to (A,B)$ consist of sequences $w_0 - w_1 - \cdots - w_n$ of n+1 arrows in $\mathbb{C}$ separated by n gaps notated –, where each arrow must have type $w_i : B_i \to A_{i+1}$ for $0 \leq i \leq n$, under the convention that $B_0 = A$ and $A_{n+1} = B$;
- composition of spliced arrows is performed by "splicing into the gaps" (see next slide)
- the identity operation on (A,B) is given by $id_A – id_B$.

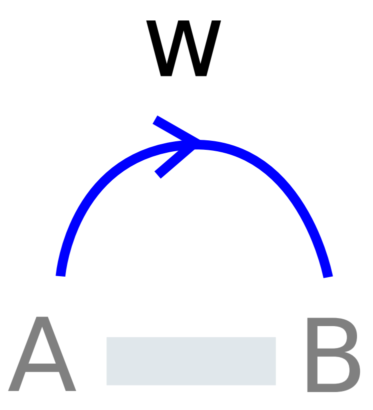
(W[$\mathbb{C}$] generalizes W[Σ], taking $\mathbb{C} = \mathbb{B}_\Sigma$ the free monoid seen as one-object category.)

# The operad of spliced arrows

*operations*



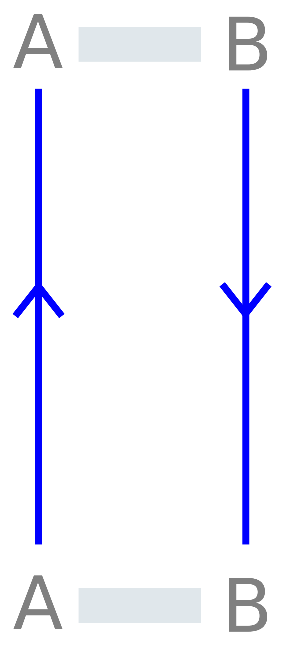

$w_0 - w_1 - w_2 - w_3 : (A_1,B_1),(A_2,B_2),(A_3,B_3) \to (A,B)$



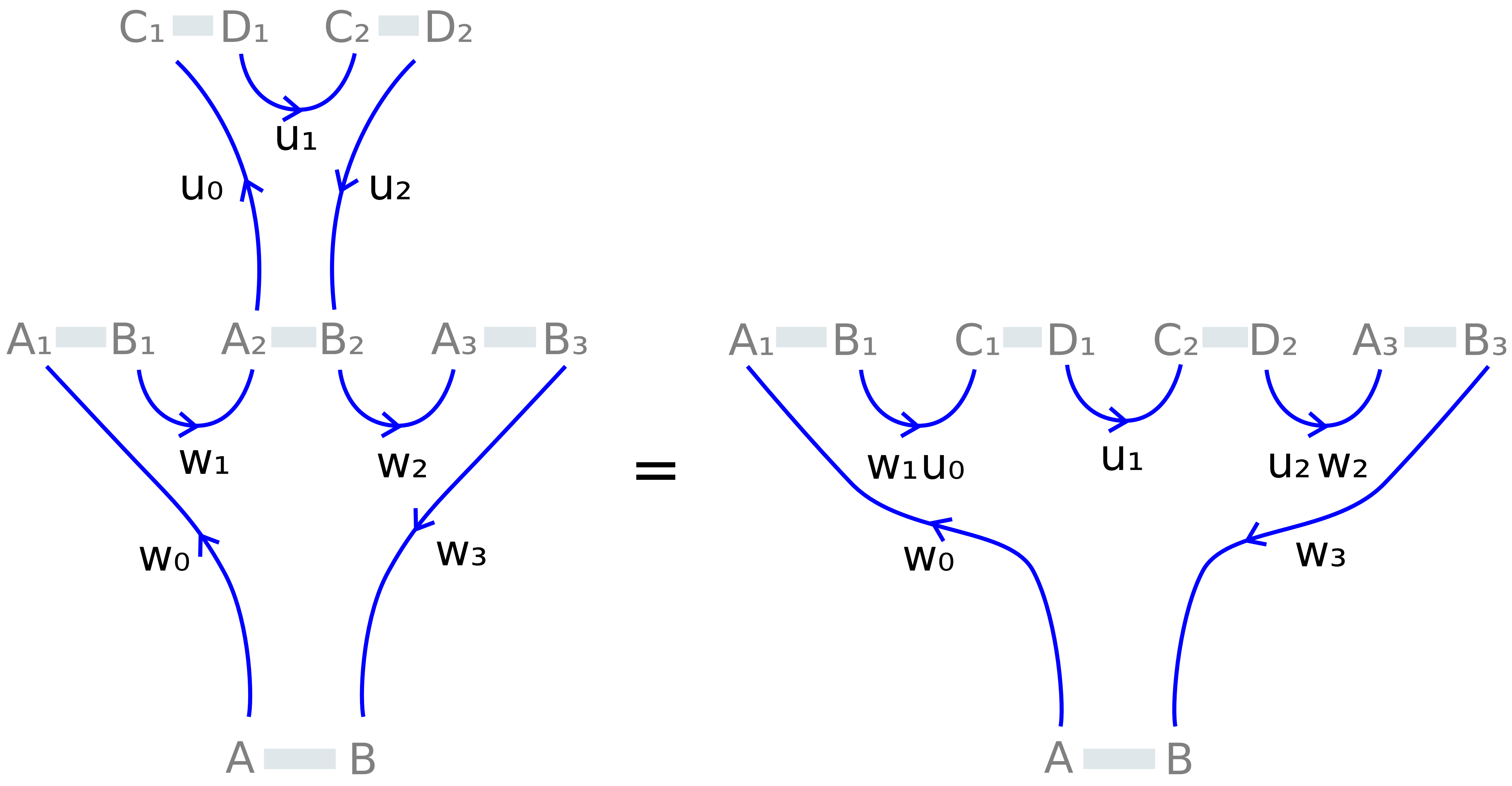

$w : (A,B)$

# The operad of spliced arrows

*identity*




*partial composition*



# The splicing functor

The operad of spliced arrows construction defines a functor

$$\mathsf{Cat} \xrightarrow{\mathsf{W}[\text{-}]} \mathsf{Operad}$$

since any functor of categories $F : \mathbb{C} \to \mathbb{D}$ induces a functor of operads $\mathsf{W}[F] : \mathsf{W}[\mathbb{C}] \to \mathsf{W}[\mathbb{D}]$.

# Species (some terminology)

A (colored non-symmetric) **species** is a span of sets of the form

$$C^* \xleftarrow{\;i\;} V \xrightarrow{\;o\;} C$$

with the following interpretation: C is a set of "colors", V a set of "nodes", and $i : V \to C^*$ and $o : V \to C$ return respectively the list of input colors and the unique output color of each node. We say a species is **finite** (aka "polynomial") iff both C and V are finite. A **map of species** is a pair of functions $(\varphi_C,\varphi_V)$ making the diagram commute:

$$\begin{array}{ccccc} C^* & \xleftarrow{\;i\;} & V & \xrightarrow{\;o\;} & C \\ \downarrow{\scriptstyle \varphi_C^*} & & \downarrow{\scriptstyle \varphi_V} & & \downarrow{\scriptstyle \varphi_C} \\ D^* & \xleftarrow{\;i'\;} & W & \xrightarrow{\;o'\;} & D \end{array}$$

# The free / forgetful adjunction

Any operad has an **underlying species**, where C is the set of colors and V the set of operations, just forgetting about composition and identity.

Conversely, to any species $\mathbb{S}$ there is an associated **free operad** Free $\mathbb{S}$.

$$\text{Species} \underset{\text{Forget}}{\overset{\text{Free}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \text{Operad}$$

$$\text{Species}(\text{Free } \mathbb{S}, \mathbb{O}) \cong \text{Operad}(\mathbb{S}, \text{Forget } \mathbb{O})$$

# Definition

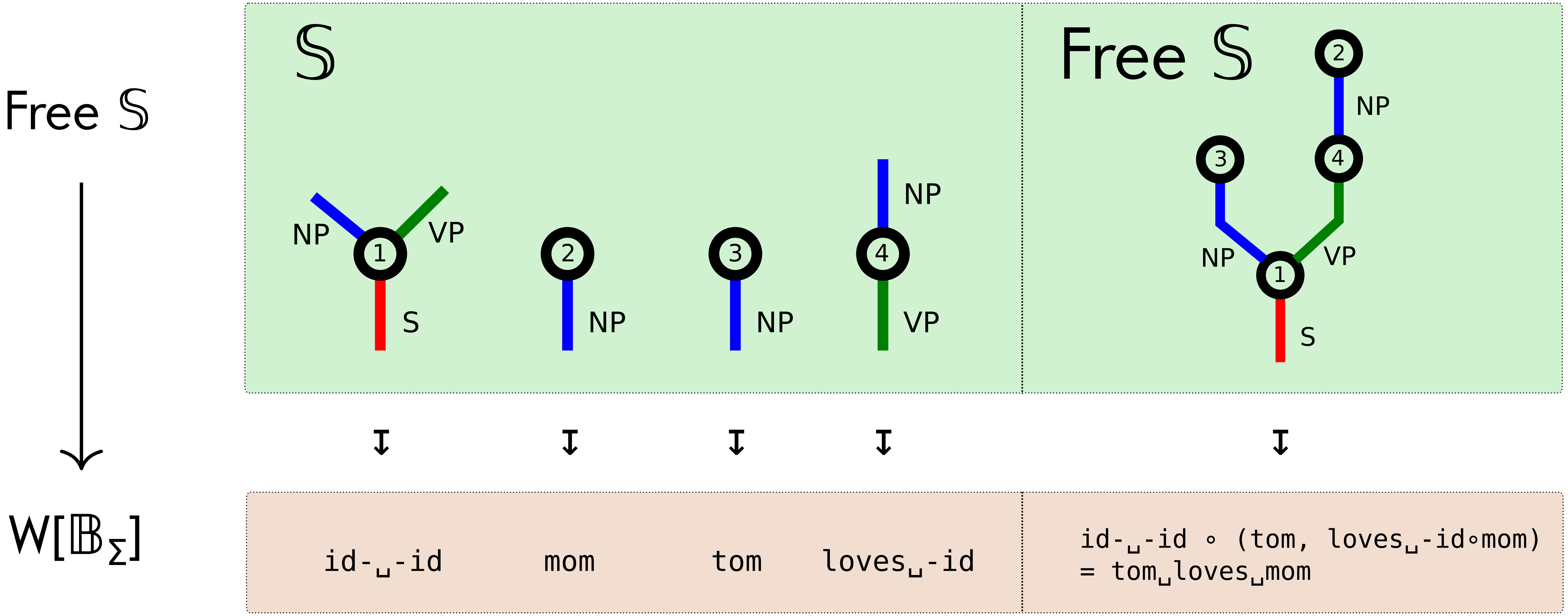
A **context-free grammar of arrows** is a tuple $G = (\mathbb{C}, \mathbb{S}, S, \varphi)$ consisting of a category $\mathbb{C}$, a finite species $\mathbb{S}$ equipped with a distinguished color $S \in \mathbb{S}$ and a functor of operads $p : \text{Free } \mathbb{S} \to W[\mathbb{C}]$.

The **context-free language of arrows** $L_G$ generated by the grammar G is the subset of arrows in $\mathbb{C}$ which, seen as constants of $W[\mathbb{C}]$, are in the image of constants of color S in Free $\mathbb{S}$, that is, $L_G = \{ p(\alpha) \mid \alpha : S \}$.

Proposition: A language $L \subseteq \Sigma^*$ is context-free in the classical sense iff it is the language of arrows of a context-free grammar over $\mathbb{B}_\Sigma$.

# (Another look at the example)

```
1 : S → NP VP
2 : NP → mom
3 : NP → tom
4 : VP → loves NP
```



# Refining classical CFGs with "gap types"

A feature of the general notion of CFG of arrows is that non-terminals are *sorted*. Adopting our conventions for type refinement, we sometimes write $R \sqsubset (A,B)$ to indicate $p(R) = (A,B)$ and say that R refines the **gap type** (A,B). The language generated by a grammar with start symbol $S \sqsubset (A,B)$ is a subset of $\mathbb{C}(A,B)$.

As a simple example, consider the category $\mathbb{B}_\Sigma^\top = \mathbb{B}_\Sigma +_\sigma 1$ constructed from $\mathbb{B}_\Sigma$ by freely adjoining an object $\top$ and an arrow $\$ : * \to \top$. A CFG over $\mathbb{B}_\Sigma^\top$ may include production rules that can only be applied upon reaching *end of input,* like Knuth's "0th production" rule $S' \to S\$$ from the original paper on LR parsing. (Here $S \sqsubset (*,*)$ is "classical" and $S' \sqsubset (*,\top)$ is "end-of-input-aware".)

More significant examples coming up, including CFGs over runs of automata!

# Reformulating standard properties of CFGs

Let $G = (\mathbb{C}, \mathbb{S}, S, p)$ be a CFG of arrows.

- G is **linear** iff $\mathbb{S}$ only has nodes of arity $\leq 1$. It is **left-linear** iff it is linear and every unary node x of $\mathbb{S}$ is mapped by p to an operation of the form id–w.
- G is **bilinear** (a generalization of Chomsky NF) iff $\mathbb{S}$ only has nodes of arity $\leq 2$.
- G is **unambiguous** iff for any constants $\alpha, \beta : S$ in Free $\mathbb{S}$, if $p(\alpha) = p(\beta)$ then $\alpha = \beta$.
- A non-terminal R is **nullable** if there exists a constant $\alpha : R$ of Free $\mathbb{S}$ s.t. $p(\alpha) = id$.
- A non-terminal R is **useful** if there exists a constant $\alpha : R$ and a unary op $\beta : R \to S$. Note that if G has no useless non-terminals then G is unambiguous iff p is faithful.

# Basic closure properties of CF languages

**[Union]** If $L_1$, $L_2 \subseteq \mathbb{C}(A,B)$ are CF, so is $L_1 \cup L_2 \subseteq \mathbb{C}(A,B)$.

**[Spliced concatenation]** If $L_1 \subseteq \mathbb{C}(A_1,B_1),\ldots,L_n \subseteq \mathbb{C}(A_n,B_n)$ are CF, and $w_0 - w_1 - \cdots - w_n : (A_1,B_1),\ldots,(A_n,B_n) \to (A,B)$ is an operation of $W[\mathbb{C}]$, then $w_0 L_1 w_1 \cdots L_n w_n \subseteq \mathbb{C}(A, B)$ is also CF.

**[Functorial image]** If $L \subseteq \mathbb{C}(A, B)$ is CF, and $F : \mathbb{C} \to \mathbb{D}$ is a functor of categories, then $F(L) \subseteq \mathbb{D}(F(A), F(B))$ is also CF.

(Proofs left as an exercise!)

## The translation principle

Let $G_1 = (\mathbb{C}, \mathbb{S}_1, S_1, p_1)$ and $G_2 = (\mathbb{C}, \mathbb{S}_2, S_2, p_2)$ be two CFGs over the same category $\mathbb{C}$.

If there is a fully faithful functor of operads $T : \text{Free } \mathbb{S}_1 \to \text{Free } \mathbb{S}_2$ such that $p_1 = T\, p_2$ and $T(S_1) = S_2$, then $L_{G_1} = L_{G_2}$.

Example use of translation principle: *for any CFG of arrows, there is a bilinear CFG of arrows generating the same language.*

# Parsing as a lifting problem

Besides characterizing the language generated by a grammar, we're often interested in the dual problem of parsing. In our functorial formulation of context-free grammars, parsing an arrow w may be considered as the problem of computing its inverse image along $p : \mathrm{Free}\ \mathbb{S} \to W[\mathbb{C}]$.

One high-level tool for analyzing this problem is the correspondence between functors of categories $p : \mathbb{D} \to \mathbb{T}$ and lax functors $F : \mathbb{T} \to \mathrm{Span(Set)}$ into the bicategory of spans of sets, which can be extended smoothly to functors of operads. Adapting terminology introduced by Ahrens and Lumsdaine, we refer to a lax functor of operads $F : \mathbb{T} \to \mathrm{Span(Set)}$ as a **displayed operad**.

# Displayed free operads, and generalized CYK parsing

One can derive an inductive formula for displayed free operads, which refines the inductive formula for free operads $\text{Free}\ \mathbb{S} \cong I + \mathbb{S} \circ \text{Free}\ \mathbb{S}$ that characterizes the free operad over $\mathbb{S}$ as a species of $\mathbb{S}$-labelled trees.

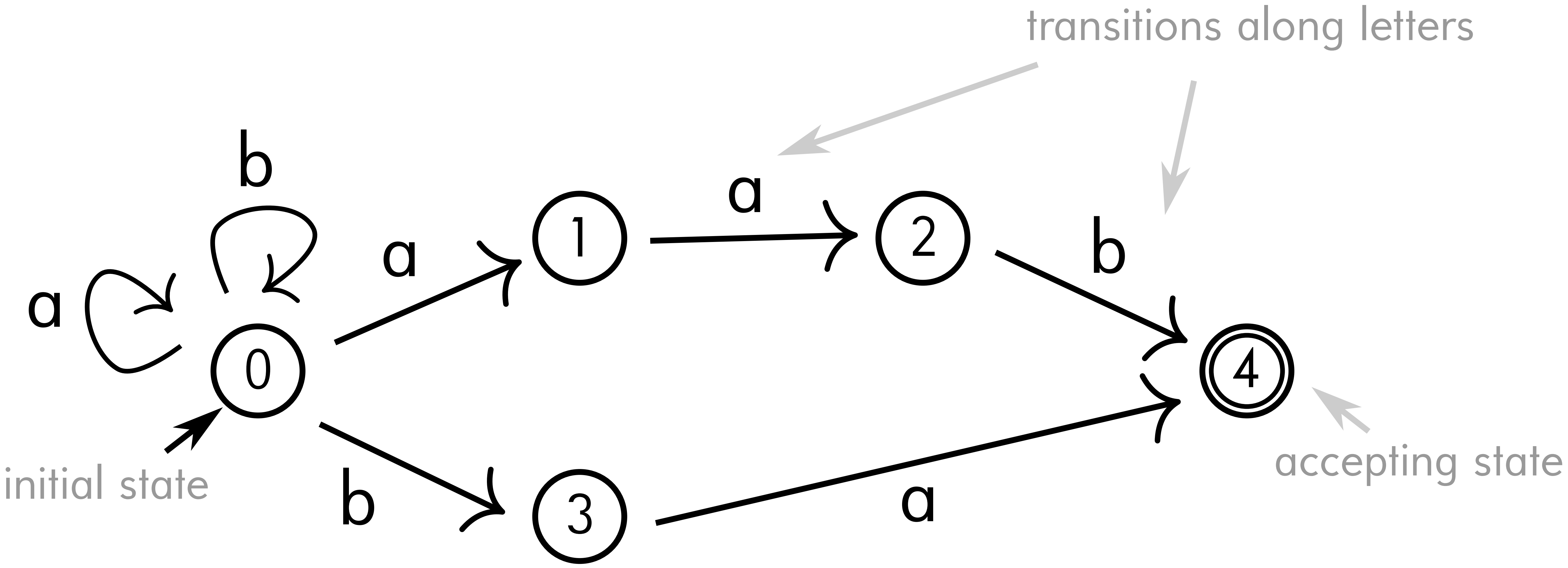
Specializing the formula to the underlying functor of a CFG seen as a displayed operad $F : W[\mathbb{C}] \to \text{Span}(\text{Set})$, we obtain a formula for the fiber $F_w$ of parse trees of any given arrow w. We can also derive an inductive rule for computing the set $N_w$ of non-terminals deriving w, which is essentially the rule given by Leermakers (1989) in his generalization of CYK parsing to arbitrary CFGs. As he explained, the relation $N_w$ can be solved in cubic time for bilinear grammars.

$$\frac{w = w_0 u_1 w_1 \dots u_k w_k \qquad \begin{array}{c}(x : R_1, \dots, R_k \to R) \in \mathcal{S} \\ \phi(x) = w_0 - w_1 - \dots - w_n\end{array} \qquad R_1 \in N_{u_1} \quad \dots \quad R_k \in N_{u_k}}{R \in N_w}$$

# 3. Finite-state automata over categories and operads

# Reminder on finite state automata

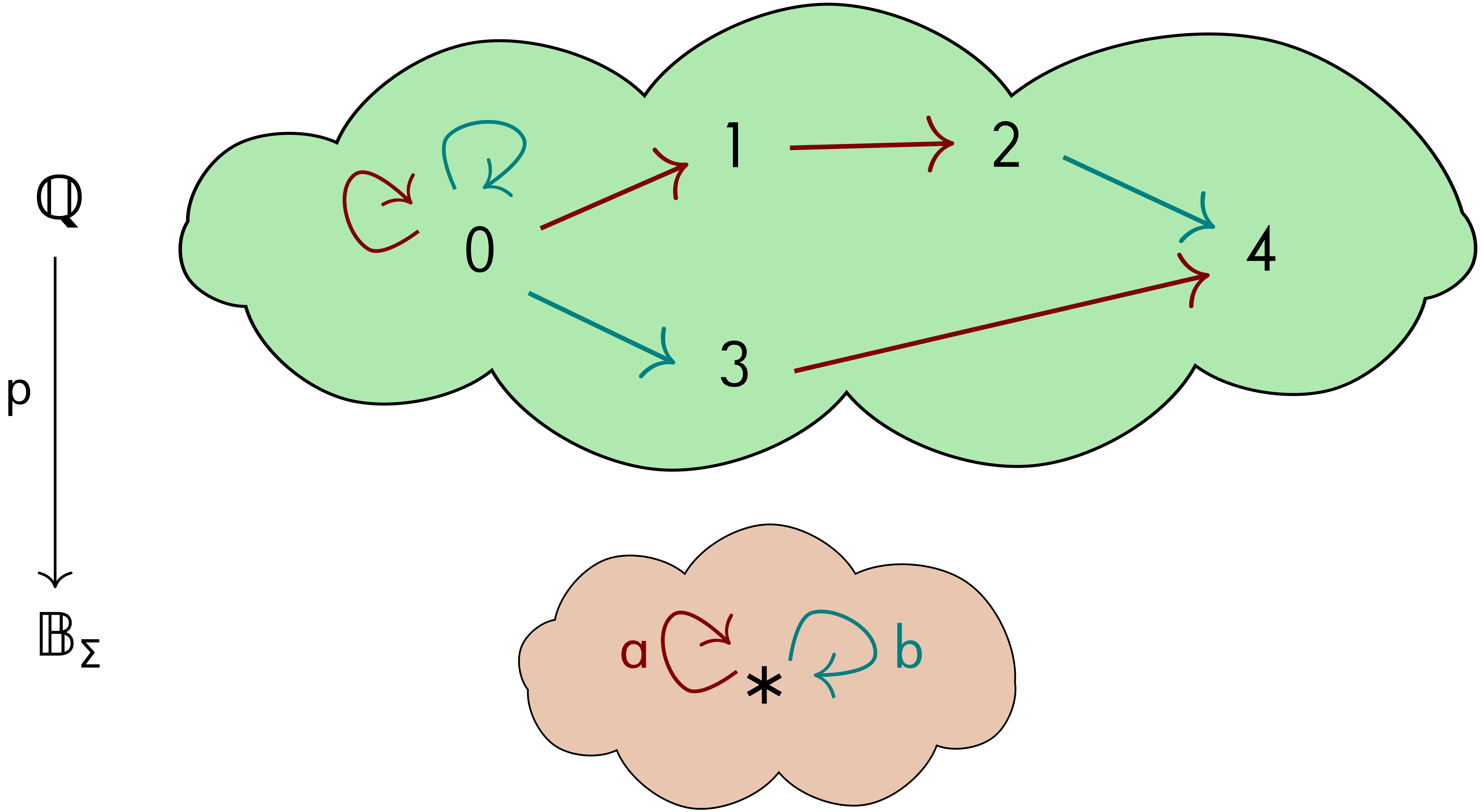
An **NDFA:** [recognizing the language (a+b)*(abb+ba)]




***alphabet*** $\Sigma = \{a,b\}$ ***state set*** $Q = \{0,1,2,3,4\}$

*(no $\varepsilon$-transitions)*

# Representing automata as functors



# Two key properties of NDFAs

Let $p : \mathbb{D} \to \mathbb{T}$ be a functor of categories.

p is **finitary** if either of the following equivalent conditions hold:

- the fibers $p^{-1}(A)$ and $p^{-1}(w)$ are finite for every object A and arrow w in $\mathbb{T}$;
- the associated lax functor $F : \mathbb{T} \to \mathrm{Span(Set)}$ factors via Span(FinSet).

ULF = "unique lifting of factorizations" (Lawvere & Meni)

p is **ULF** if either of the following equivalent conditions hold:

- for any arrow $\alpha$ of $\mathbb{D}$, if $p(\alpha) = uv$ for some pair of arrows u and v of $\mathbb{T}$, there exists a unique pair of arrows $\beta$ and $\gamma$ in $\mathbb{D}$ such that $\alpha = \beta\gamma$, $p(\beta) = u$, $p(\gamma) = v$.
- the associated lax functor $F : \mathbb{T} \to \mathrm{Span(Set)}$ is a pseudofunctor.

Proposition: a functor $p : \mathbb{Q} \to \mathbb{B}_\Sigma$ is the underlying bare automaton of a NDFA with alphabet $\Sigma$ iff p is both finitary and ULF.

# Definition

A **NDFA over a category** is a tuple $M = (\mathbb{C}, \mathbb{Q}, p : \mathbb{Q} \to \mathbb{C}, q_0, q_f)$ consisting of two categories $\mathbb{C}$ and $\mathbb{Q}$, a finitary ULF functor $p : \mathbb{Q} \to \mathbb{C}$, and a pair $q_0, q_f$ of objects of $\mathbb{Q}$.

The **regular language of arrows** $L_M$ recognized by the automaton M is the subset of arrows in $\mathbb{C}$ that can be lifted along p to an arrow $\alpha : q_0 \to q_f$ in $\mathbb{Q}$, that is, $L_M = \{ p(\alpha) \mid \alpha : q_0 \to q_f \}$.

Proposition: A language $L \subseteq \Sigma^*$ is regular in the classical sense iff L\$ is the regular language of arrows of a NDFA over $\mathbb{B}_\Sigma^\top$.

# Automata over operads

The notions of finitary and ULF extend smoothly to functors of operads.

By analogy, an **NDFA over an operad** is a tuple $M = (\mathbb{O}, \mathbb{Q}, p : \mathbb{Q} \to \mathbb{O}, q)$ where $p : \mathbb{Q} \to \mathbb{O}$ is a finitary ULF functor of operads, and q a color of $\mathbb{Q}$.

When $\mathbb{O}$ is a free operad, this recovers the standard notion of ND finite state tree automaton. But the notion of NDFA over an operad is more general!

Proposition: if a functor of categories $p : \mathbb{Q} \to \mathbb{C}$ is finitary or ULF, so is the functor of operads $W[p] : W[\mathbb{Q}] \to W[\mathbb{C}]$.

∴ *any NDFA over a category induces an NDFA over its spliced arrow operad,* by the mapping $(p : \mathbb{Q} \to \mathbb{C}, q_0, q_f) \mapsto (W[p] : W[\mathbb{Q}] \to W[\mathbb{C}], (q_0,q_f))$

# 4. The Representation Theorem

# Overview

Chomsky & Schützenberger (1963): Any CF language is the homomorphic image of the intersection of a Dyck language with a regular language.

Our version: Any CF language of arrows in $\mathbb{C}$ is the functorial image of the intersection of a $\mathbb{C}$-chromatic tree contour language and a regular language.

The proof relies on two constructions that are of more general interest:

1. The pullback of a CFG of arrows along an NDFA, which we use to show that CF languages are closed under intersection with regular languages.

2. The *contour category* of an operad, providing a left adjoint to the splicing functor, which we use to define a "universal CFG" for any pointed finite species.

# An important property of ULF functors

Let $p_Q : \mathbb{Q} \to \mathbb{O}$ be a ULF functor of operads. Then the pullback of $p : \text{Free}\ \mathbb{S} \to \mathbb{O}$ along $p_Q$ in the category of operads is obtained from a corresponding pullback of $\varphi : \mathbb{S} \to \mathbb{O}$ along $p_Q : \mathbb{Q} \to \mathbb{O}$ in Species.

$$\begin{array}{ccc} \text{Free}\ \mathbb{S}' & \xrightarrow{\text{Free}\ \psi} & \text{Free}\ \mathbb{S} \\ {\scriptstyle p'}\downarrow & \textit{pullback} & \downarrow{\scriptstyle p} \\ \mathbb{Q} & \xrightarrow[p_Q]{} & \mathbb{O} \end{array} \qquad \Bigg| \qquad \begin{array}{ccc} \mathbb{S}' & \xrightarrow{\psi} & \mathbb{S} \\ {\scriptstyle \varphi'}\downarrow & \textit{pullback} & \downarrow{\scriptstyle \varphi} \\ \mathbb{Q} & \xrightarrow[p_Q]{} & \mathbb{O} \end{array}$$

# Pulling back a CFG along a NDFA

By the previous result, we can compute the pullback on the right:

$$
\begin{array}{ccc}
\mathrm{Free}\ \mathbb{S}' & \xrightarrow{\mathrm{Free}\ \psi} & \mathrm{Free}\ \mathbb{S} \\
{\scriptstyle p'}\downarrow & \textit{pullback} & \downarrow{\scriptstyle p_G} \\
W[\mathbb{Q}] & \xrightarrow[W[p_M]]{} & W[\mathbb{C}]
\end{array}
\qquad\Bigg|\qquad
\begin{array}{ccc}
\mathbb{S}' & \xrightarrow{\psi} & \mathbb{S} \\
{\scriptstyle \varphi'}\downarrow & \textit{pullback} & \downarrow{\scriptstyle \varphi_G} \\
W[\mathbb{Q}] & \xrightarrow[W[p_M]]{} & W[\mathbb{C}]
\end{array}
$$

The pullback of G along M is the grammar $G' = (\mathbb{Q}, \mathbb{S}', (S,(q_0,q_f)), p')$.
Note that $G'$ generates a language of runs of M!

Taking the image of $G'$ along $p_M$ yields a grammar generating $L_G \cap L_M$.

# The contour category of an operad

Let $\mathbb{O}$ be an operad. The category $C[\mathbb{O}]$ is a quotient of the free category with:

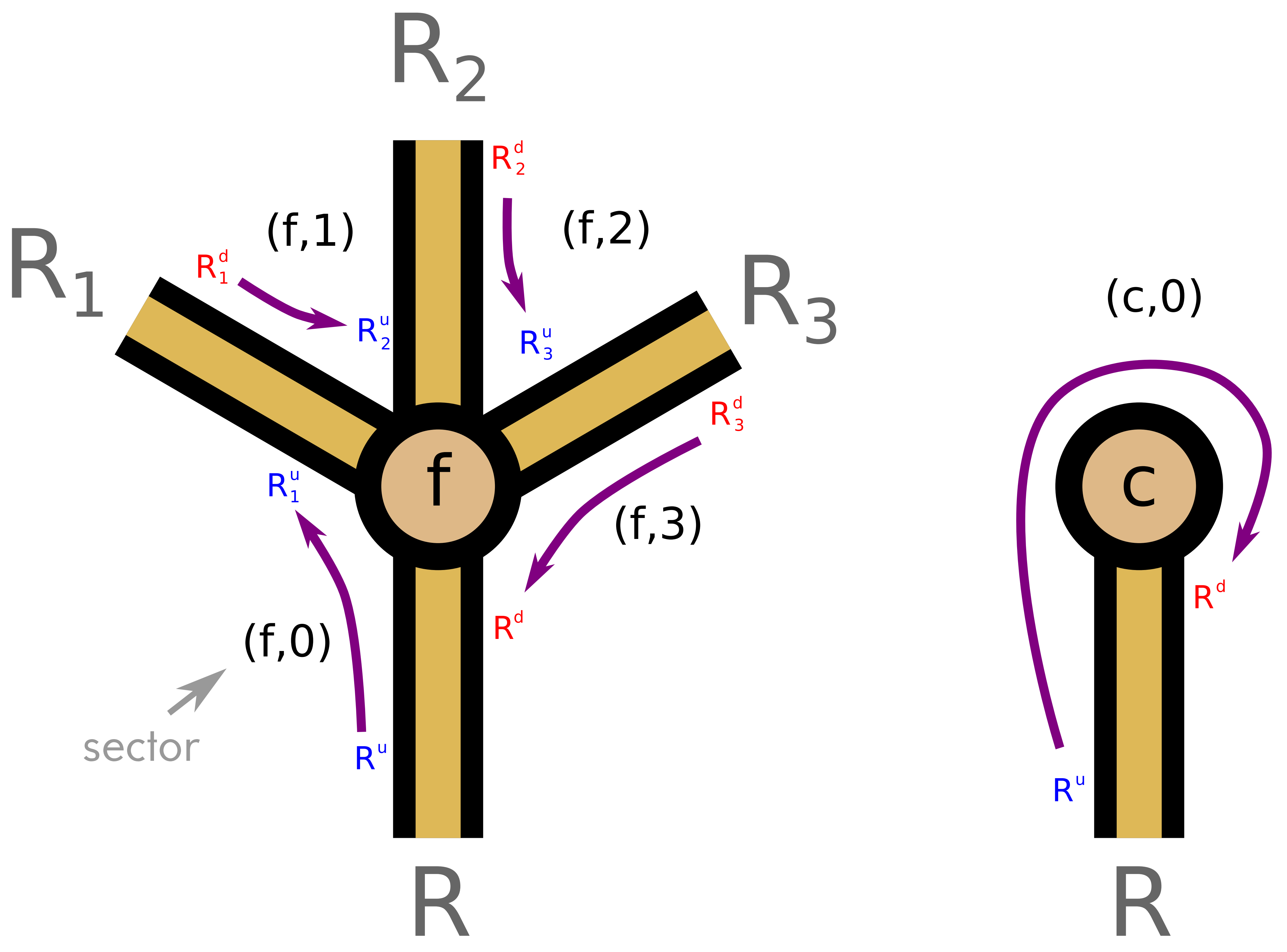
- objects given by *oriented colors* $R^\varepsilon$ consisting of a color R of $\mathbb{O}$ and an orientation $\varepsilon \in \{ u,d \}$ ("up" or "down");
- arrows generated by pairs (f,i) of an operation $f : R_1,\dots,R_n \to R$ of $\mathbb{O}$ and an index $0 \le i \le n$, defining an arrow $R_i^d \to R_{i+1}^u$ where $R_0^d = R^u$ and $R_{n+1}^u = R^d$;

subject to the equations $id_{R^u} = (id_R,0)$ and $id_{R^d} = (id_R,1)$ plus the equations

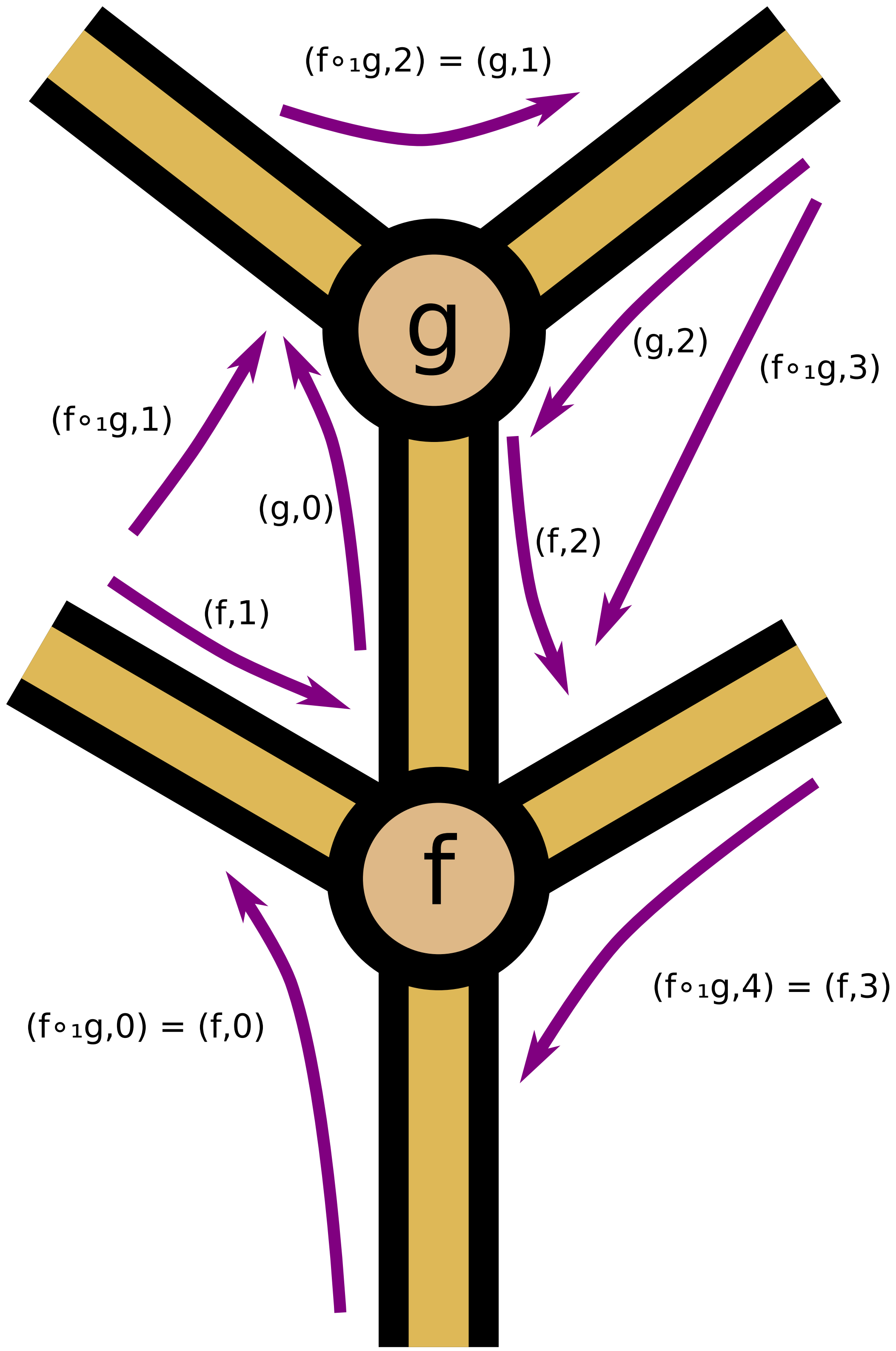
$$(f \circ_i g, j) = \begin{cases} (f, j) & j < i \\ (f, i)(g, 0) & j = i \\ (g, j-i) & i < j < i+m \\ (g, m)(f, i+1) & j = i+m \\ (f, j-m+1) & j > i+m \end{cases} \qquad (f \circ_i c, j) = \begin{cases} (f, j) & j < i \\ (f, i)(c, 0)(f, i+1) & j = i \\ (f, j+1) & j > i \end{cases}$$

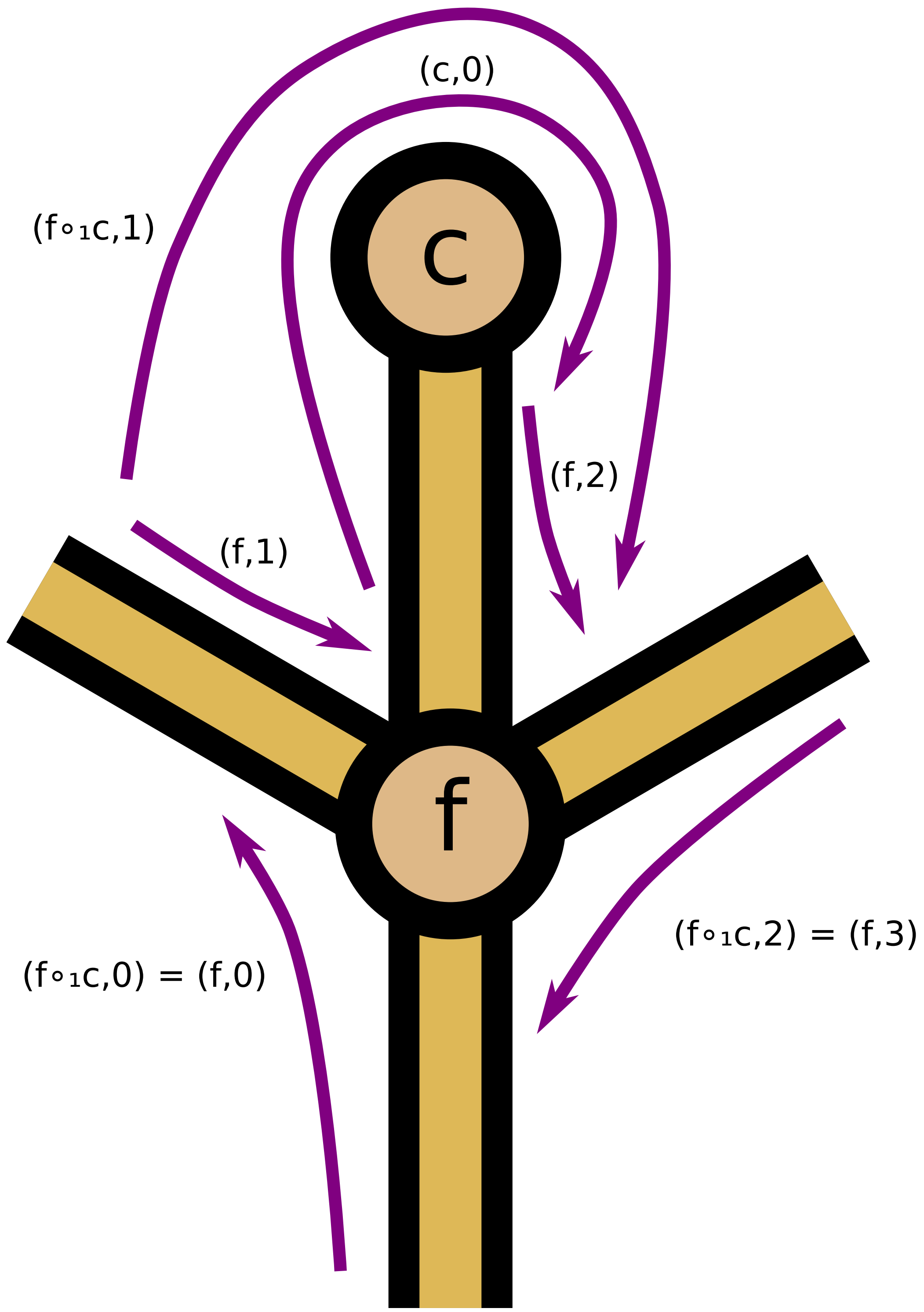
for every operation f, operation g of positive arity m > 0, and constant c.

# The contour category of an operad



# The contour category of an operad



# The contour / splicing adjunction

This construction provides a left adjoint to the splicing contruction:

$$\mathrm{Operad} \underset{W[-]}{\overset{C[-]}{\rightleftarrows}} \mathrm{Cat} \quad (\bot)$$

$$\mathrm{Operad}(\mathbb{O}, W[\mathbb{C}]) \cong \mathrm{Cat}(C[\mathbb{O}], \mathbb{C})$$

The unit and counit have nice descriptions:

$$\eta : \mathbb{O} \to W[C[\mathbb{O}]] \qquad \varepsilon : C[W[\mathbb{C}]] \to \mathbb{C}$$

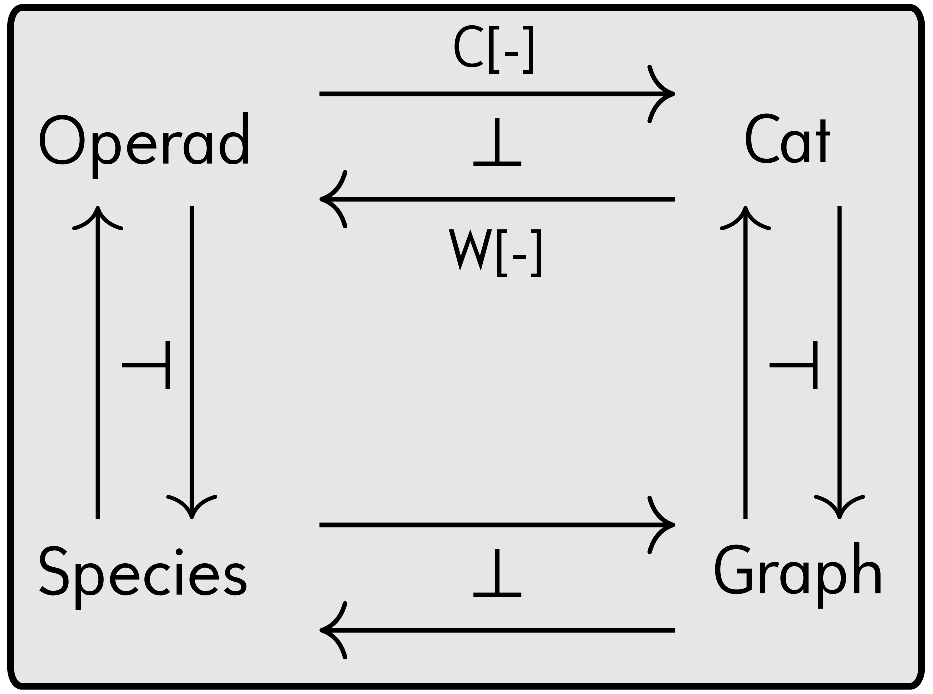
$$R \mapsto (R^u, R^d) \qquad (A,B)^u \mapsto A \quad (A,B)^d \mapsto B$$

$$f \mapsto (f,0)-\cdots-(f,n) \qquad (w_0-\cdots-w_n, i) \mapsto w_i$$

# Free contour categories

The contour category of a free operad is itself a free category, with C[Free 𝕊] generated by the **corners*** (x,i) consisting of an n-ary node x and index $0 \leq i \leq n$.

We sometimes write C[𝕊] as another name for this category.




Although C[-] dœs not preserve ULF in general, we have that for any species map $\psi : \mathbb{S} \to \mathbb{R}$, the functor of categories $C[\psi] : C[\mathbb{S}] \to C[\mathbb{R}]$ is ULF.

*Note that the word "corner" comes from the theory of planar maps, but in parsing theory, corners are called "dotted rules"!

# The universal CFG of a pointed finite species

By the contour / splicing adjunction, any $p : \mathrm{Free}\ \mathbb{S} \to W[\mathbb{C}]$ factors as

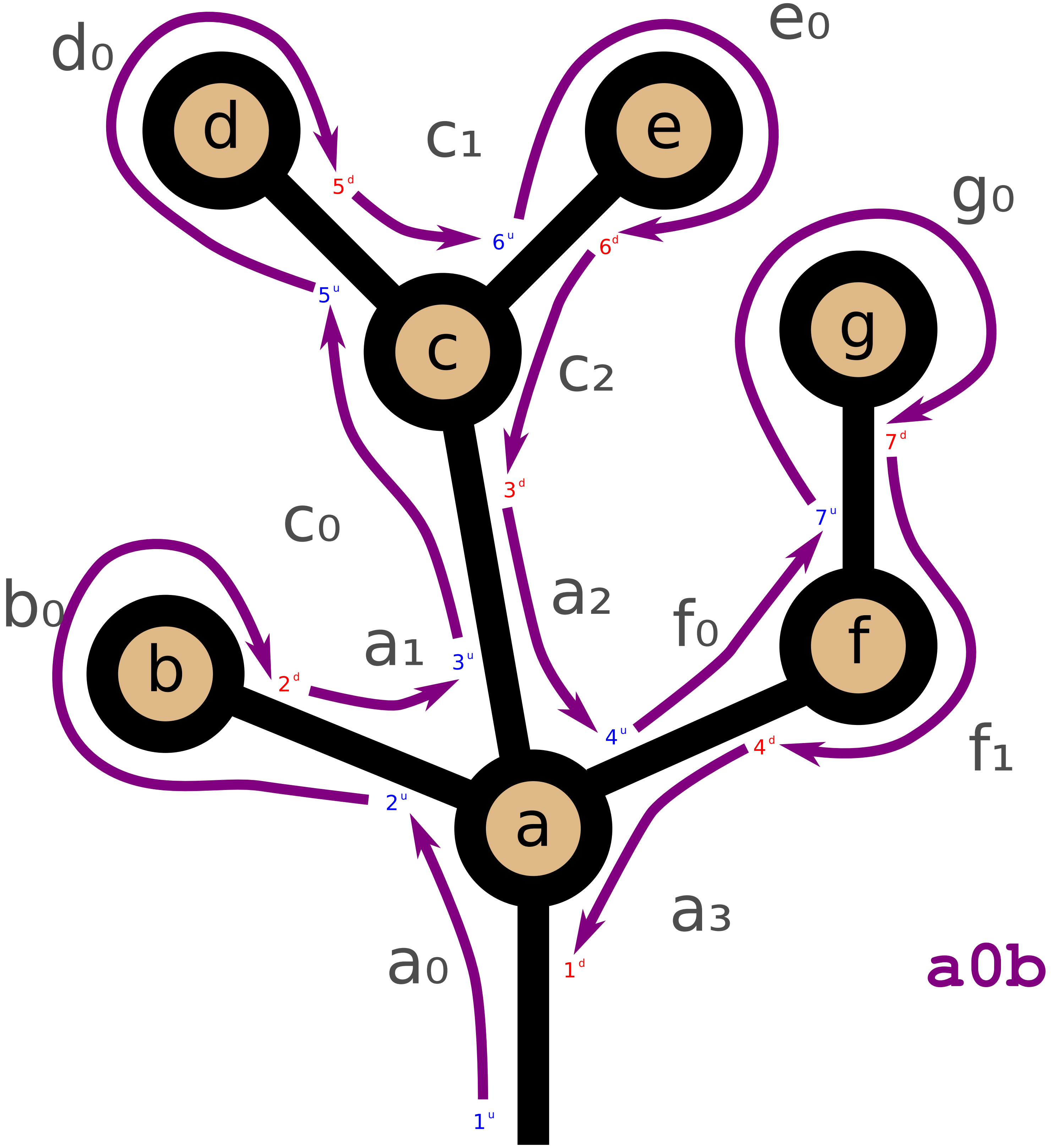
$$\mathrm{Free}\ \mathbb{S} \xrightarrow{\eta_{\mathbb{S}}} W[C[\mathrm{Free}\ \mathbb{S}]] \xrightarrow{W[q]} W[\mathbb{C}]$$

for a unique functor of categories $q : C[\mathrm{Free}\ \mathbb{S}] \to \mathbb{C}$.

The CFG $\mathrm{Univ}_{\mathbb{S},S} = (C[\mathrm{Free}\ \mathbb{S}],\mathbb{S},S,\eta_{\mathbb{S}})$ is therefore "universal", in the sense that any other CFG $G = (\mathbb{C},\mathbb{S},S,p)$ with the same species and start symbol is obtained uniquely as the functorial image $G = q\ \mathrm{Univ}_{\mathbb{S},S}$.

The language generated by $\mathrm{Univ}_{\mathbb{S},S}$ is a language of **tree contour words.**

# A tree contour word over a species $\mathbb{S}$




$\mathbb{S}$

a : 2,3,4 → 1

b : 2

c : 5,6 → 3

g : 7

d : 5

e : 6

f : 7 → 4

**a0b0a1c0d0c1e0c2a2f0g0f1a3** : $1^u \rightarrow 1^d$

# Idea of the representation theorem

Separate the generation of a CF language into three pieces:

1. generate "uncolored" contour words describing shapes of $\mathbb{S}$-trees;

2. use an automaton to check that the contour words denote well-colored $\mathbb{S}$-trees with root color S;

3. interpret each corner of the contour as an appropriate arrow.

# Another basic fact about species

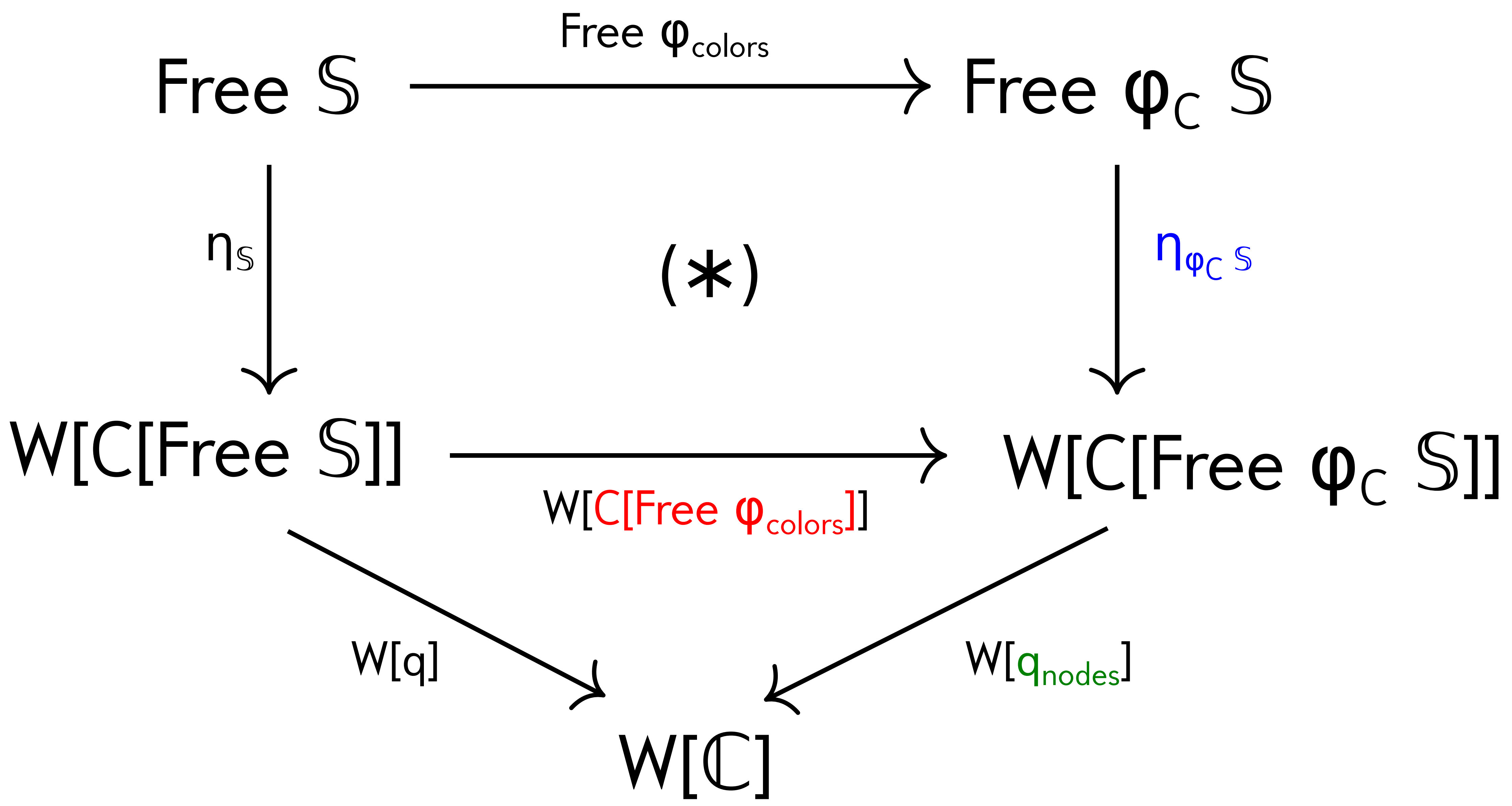
Any map of species $\varphi : \mathbb{S} \to \mathbb{R}$ factors as:

$$\mathbb{S} \xrightarrow[\text{id on nodes}]{\varphi_{\text{colors}}} \varphi_C\, \mathbb{S} \xrightarrow[\text{id on colors}]{\varphi_{\text{nodes}}} \mathbb{R}$$

In particular, we can apply this factorization to the underlying map of species $\varphi : \mathbb{S} \to W[\mathbb{C}]$ of a given CFG of arrows.

The functor $C[\varphi_{\text{colors}}] : C[\mathbb{S}] \to C[\varphi_C\, \mathbb{S}]$ paired with the states $S^u$ and $S^d$ defines an automaton on contour words!

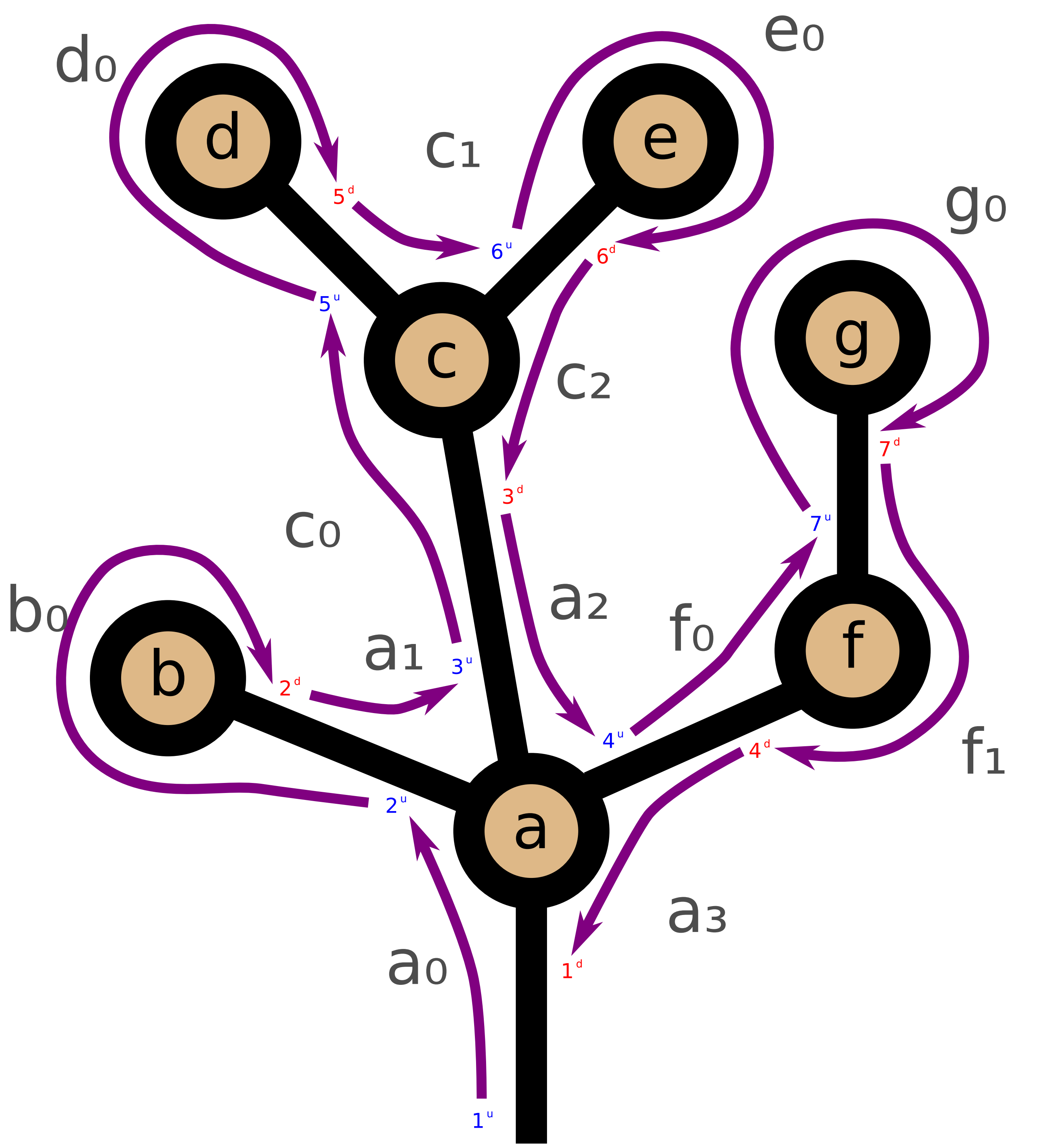
# The proof in a diagram

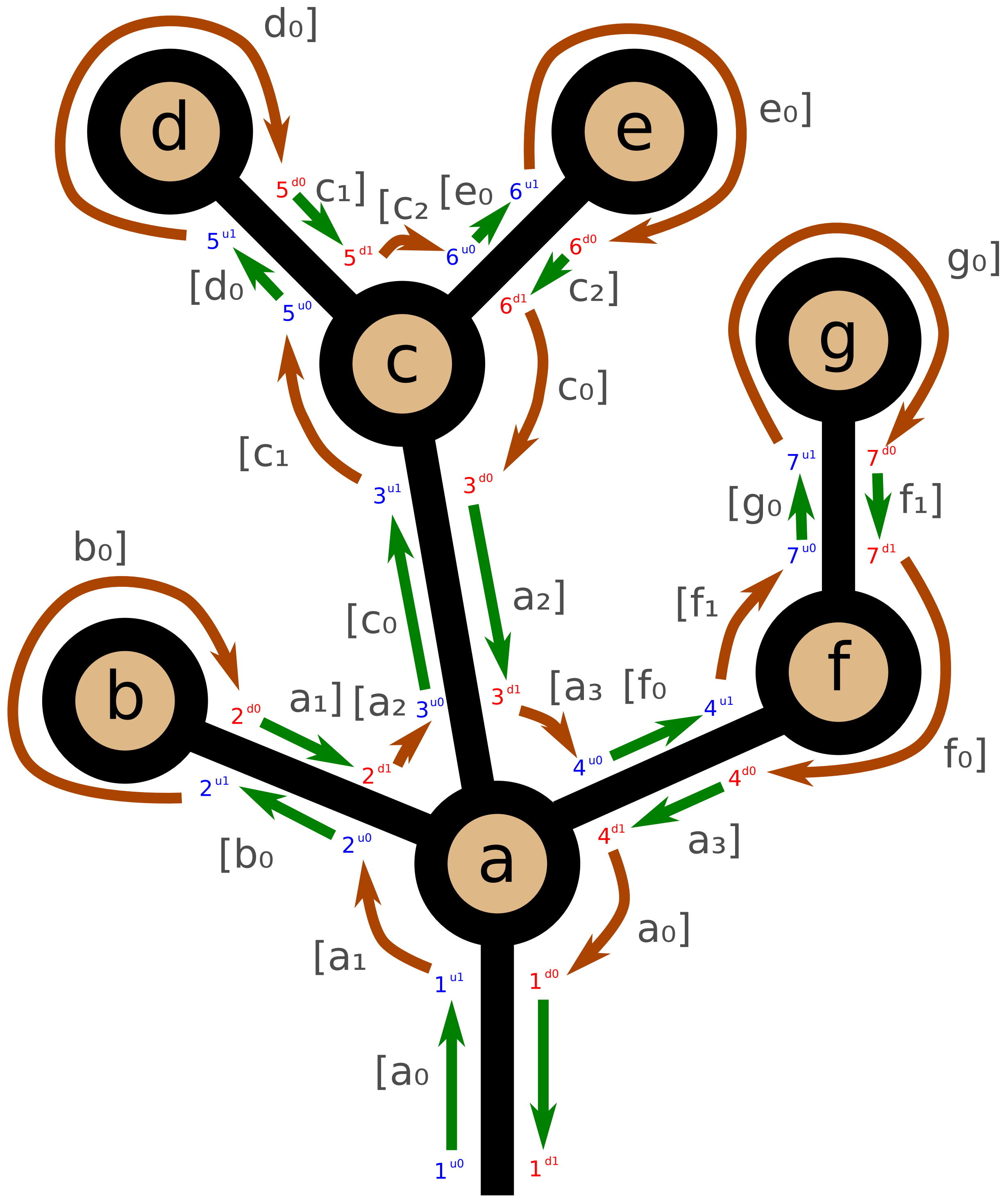



$$L_G = q\ L_{\mathbb{S},S} = q_{nodes}\ C[\varphi_{colors}]\ L_{\mathbb{S},S} = q_{nodes}\ (L_{\varphi C\mathbb{S},S} \cap L_{Mcolors})$$

*The naturality square is not a pullback, but the canonical functor Free $\mathbb{S}$ → Free $\mathbb{R}$ to the pullback is fully faithful, hence we can apply the translation principle!

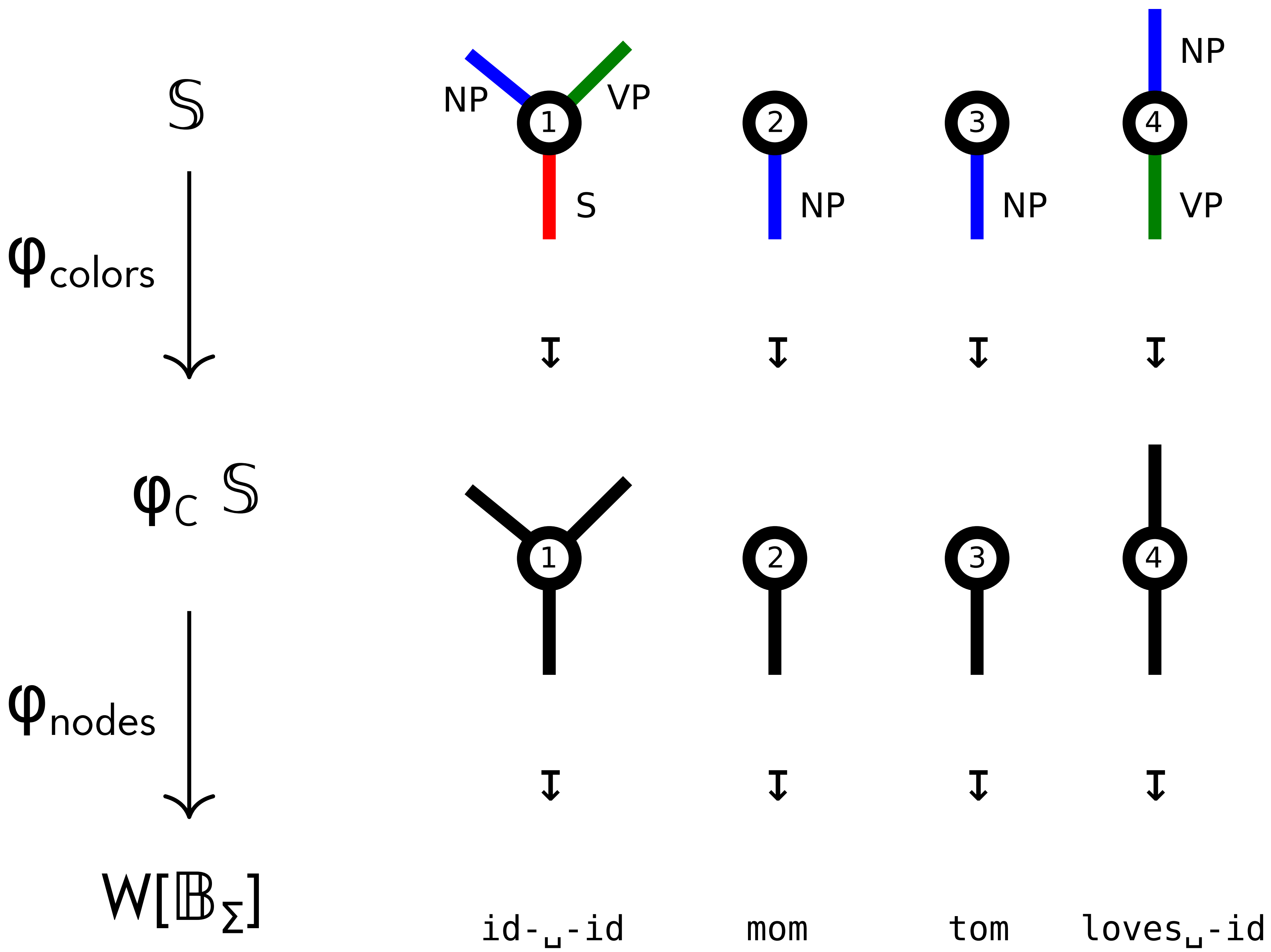
# From contour words to Dyck words



# 5. Example

# Colors / nodes factorization



# Translation of corners

$C[\varphi_C\ \mathbb{S}] \longrightarrow \mathbb{B}_\Sigma$

$1_0 \mapsto$ id
$1_1 \mapsto$ ␣
$1_2 \mapsto$ id

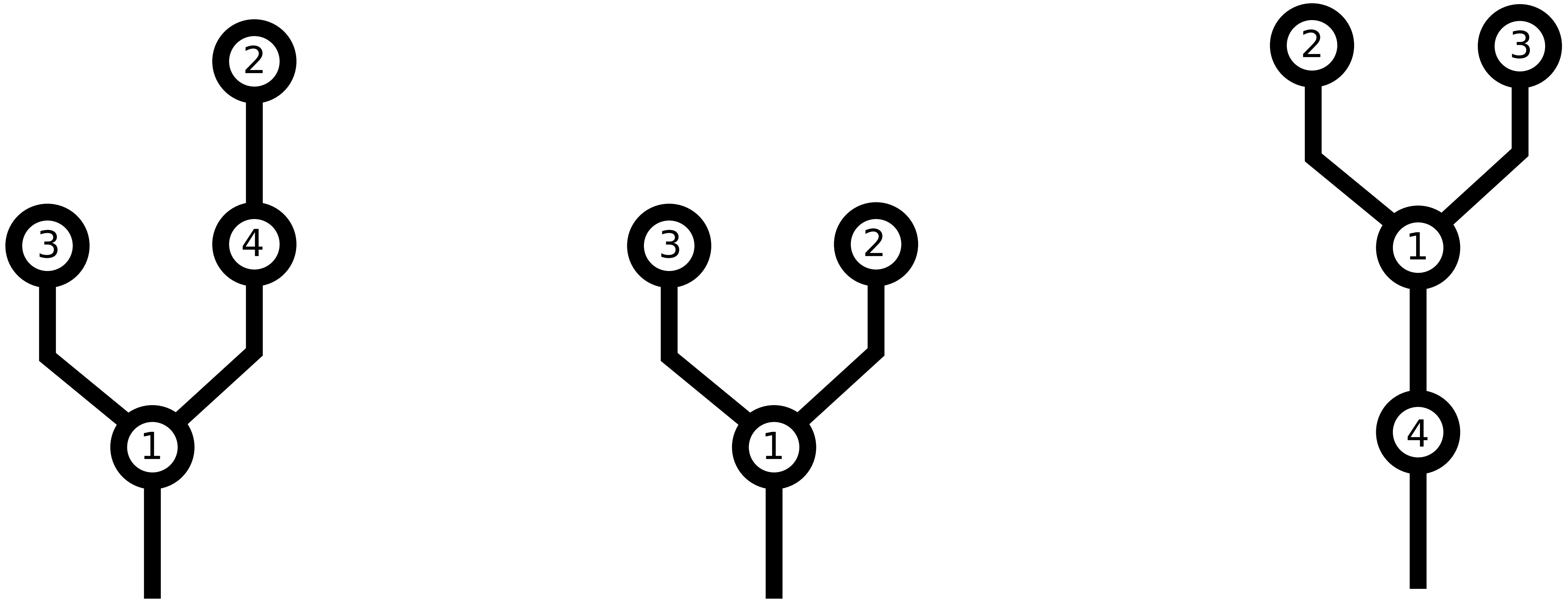
$2_0 \mapsto$ mom

$3_0 \mapsto$ tom

$4_0 \mapsto$ loves␣
$4_1 \mapsto$ id

# Uncolored tree contour words

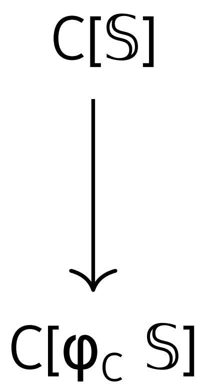
$$\mathrm{Free}\ \varphi_C\ \mathbb{S} \longrightarrow W[\mathbb{B}_\Sigma]$$

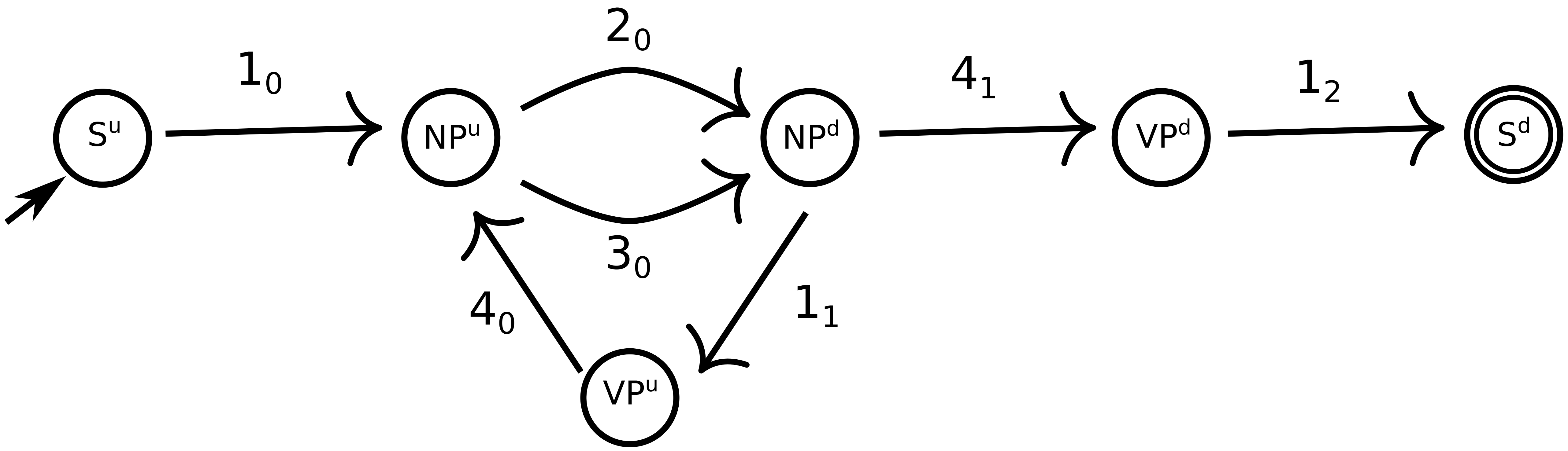


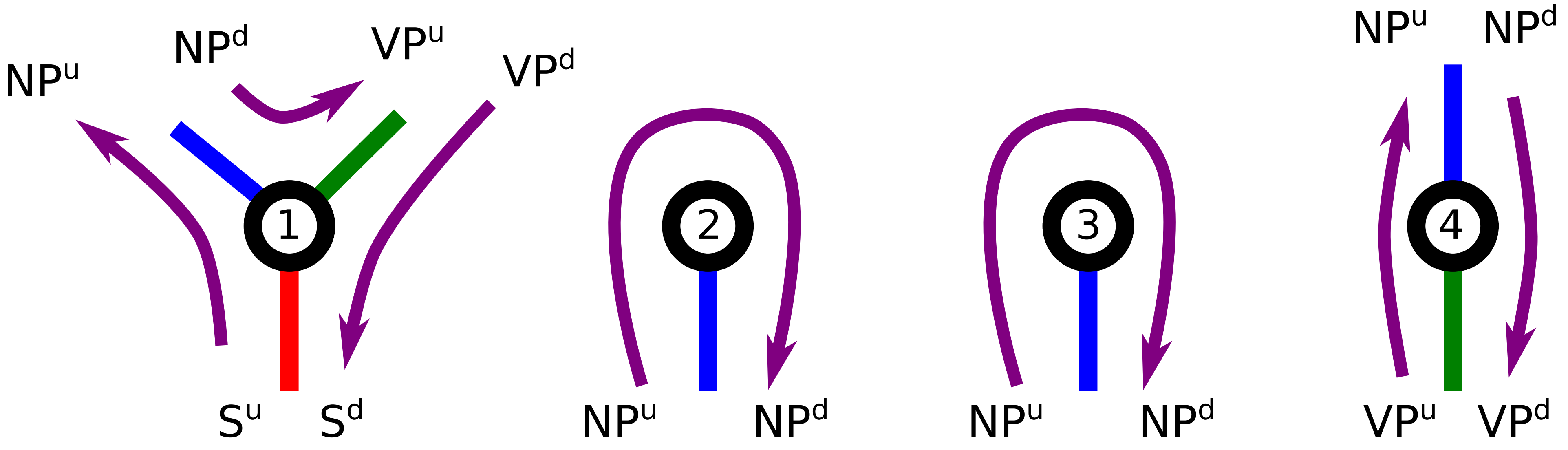

$1_0 3_0 1_1 4_0 2_0 4_1 1_2$ $\quad$ $1_0 3_0 1_1 2_0 1_2$ $\quad$ $4_0 1_0 2_0 1_1 3_0 1_2 4_1$ $\quad$ …

tom␣loves␣mom $\quad$ tom␣mom $\quad$ loves␣mom␣tom

# Coloring automaton



# 6. Conclusion

# Summary and future directions

Both CFGs and NDFAs may be naturally represented as functors, and generalized to define context-free / regular languages of arrows in a category.

Parsing may be naturally formulated as a lifting problem.

The Chomsky-Schützenberger Representation Theorem is deeply related to an elementary "contour / splicing" adjunction between operads and categories.

Are there other applications of spliced arrow operads and contour categories?

Next on our agenda: pushdown automata and LR parsing!