A connection between lambda calculus and maps

Noam Zeilberger[1]

MSR-Inria Joint Centre

OBT Workshop, Mumbai
18 January 2015

[1]based on a joint article with Alain Giorgetti: arxiv.org/abs/1408.5028

x

1

$$x(\lambda y.y) \qquad \lambda y.yx$$

2

$x(\lambda y.y(\lambda z.z))$	$x(\lambda y.\lambda z.zy)$	$(x(\lambda y.y))(\lambda z.z)$
$\lambda y.y(x(\lambda z.z))$	$\lambda y.y(\lambda z.zx)$	$\lambda y.(y(\lambda z.z))x$
$\lambda y.(yx)(\lambda z.z)$	$\lambda y.\lambda z.z(yx)$	$\lambda y.\lambda z.(zy)x$

9

1. $x(\lambda y.y(\lambda z.z(\lambda w.w)))$
2. $x(\lambda y.y(\lambda z.\lambda w.wz))$
3. $x(\lambda y.(y(\lambda z.z))(\lambda w.w))$
4. $x(\lambda y.\lambda z.z(y(\lambda w.w)))$
5. $x(\lambda y.\lambda z.z(\lambda w.wy))$
6. $x(\lambda y.\lambda z.(z(\lambda w.w))y)$
7. $x(\lambda y.\lambda z.(zy)(\lambda w.w))$
8. $x(\lambda y.\lambda z.\lambda w.w(zy))$
9. $x(\lambda y.\lambda z.\lambda w.(wz)y)$
10. $(x(\lambda y.y))(\lambda z.z(\lambda w.w))$
11. $(x(\lambda y.y))(\lambda z.\lambda w.wz)$
12. $(x(\lambda y.y(\lambda z.z)))(\lambda w.w)$
13. $(x(\lambda y.\lambda z.zy))(\lambda w.w)$
14. $((x(\lambda y.y))(\lambda z.z))(\lambda w.w)$
15. $\lambda y.y(x(\lambda z.z(\lambda w.w)))$
16. $\lambda y.y(x(\lambda z.\lambda w.wz))$
17. $\lambda y.y((x(\lambda z.z))(\lambda w.w))$
18. $\lambda y.y(\lambda z.z(x(\lambda w.w)))$
19. $\lambda y.y(\lambda z.z(\lambda w.wx))$
20. $\lambda y.y(\lambda z.(z(\lambda w.w))x)$
21. $\lambda y.y(\lambda z.(zx)(\lambda w.w))$
22. $\lambda y.y(\lambda z.\lambda w.w(zx))$
23. $\lambda y.y(\lambda z.\lambda w.(wz)x)$
24. $\lambda y.(y(\lambda z.z))(x(\lambda w.w))$
25. $\lambda y.(y(\lambda z.z))(\lambda w.wx)$
26. $\lambda y.(y(\lambda z.z(\lambda w.w)))x$
27. $\lambda y.(y(\lambda z.\lambda w.wz))x$
28. $\lambda y.((y(\lambda z.z))(\lambda w.w))x$
29. $\lambda y.(yx)(\lambda z.z(\lambda w.w))$
30. $\lambda y.(yx)(\lambda z.\lambda w.wz)$
31. $\lambda y.(y(x(\lambda z.z)))(\lambda w.w)$
32. $\lambda y.(y(\lambda z.zx))(\lambda w.w)$
33. $\lambda y.((y(\lambda z.z))x)(\lambda w.w)$
34. $\lambda y.((yx)(\lambda z.z))(\lambda w.w)$
35. $\lambda y.\lambda z.z(y(x(\lambda w.w)))$
36. $\lambda y.\lambda z.z(y(\lambda w.wx))$
37. $\lambda y.\lambda z.z((y(\lambda w.w))x)$
38. $\lambda y.\lambda z.z((yx)(\lambda w.w))$
39. $\lambda y.\lambda z.z(\lambda w.w(yx))$
40. $\lambda y.\lambda z.z(\lambda w.(wy)x)$
41. $\lambda y.\lambda z.(z(\lambda w.w))(yx)$
42. $\lambda y.\lambda z.(zy)(x(\lambda w.w))$
43. $\lambda y.\lambda z.(zy)(\lambda w.wx)$
44. $\lambda y.\lambda z.(z(y(\lambda w.w)))x$
45. $\lambda y.\lambda z.(z(\lambda w.wy))x$
46. $\lambda y.\lambda z.(z(\lambda w.w)y)x$
47. $\lambda y.\lambda z.((zy)(\lambda w.w))x$
48. $\lambda y.\lambda z.(z(yx))(\lambda w.w)$
49. $\lambda y.\lambda z.((zy)x)(\lambda w.w)$
50. $\lambda y.\lambda z.\lambda w.w(z(yx))$
51. $\lambda y.\lambda z.\lambda w.w((zy)x)$
52. $\lambda y.\lambda z.\lambda w.(wz)(yx)$
53. $\lambda y.\lambda z.\lambda w.(w(zy))x$
54. $\lambda y.\lambda z.\lambda w.((wz)y)x$

Invitation: celebrating 50 years of OEIS, 250000 sequences, and Sloane's 75th, there will be a conference at DIMACS, Rutgers, Oct 9-10 2014.

1,2,9,54,378,2916,24057 | Search | Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,2,9,54,378,2916,24057**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A000168 2*3^n*(2*n)!/(n!*(n+2)!). +20
18

(Formerly M1940 N0768)

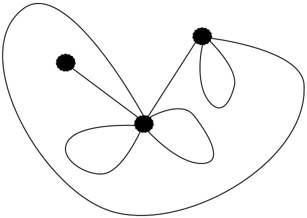
1, 2, 9, 54, 378, 2916, 24057, 208494, 1876446, 17399772, 165297834, 1602117468, 15792300756, 157923007560, 1598970451545, 16365932856990, 169114639522230, 1762352559231660, 18504701871932430, 195621134074714260, 2080697516976506220, 22254416920705240440, 239234981897581334730, 2583737804493878415084 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

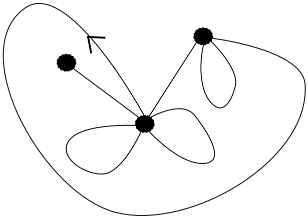
COMMENTS Number of rooted planar maps with n edges. - Don Knuth, Nov 24 2013

Number of rooted 4-regular planar maps with n vertices.

Also, number of doodles with n crossings, irrespective of the number of loops.



A planar map



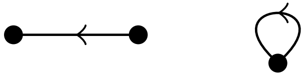
A rooted planar map

Rooted planar maps were first counted by W. T. Tutte, as part of an attack on the 4CT (which is about planar maps):

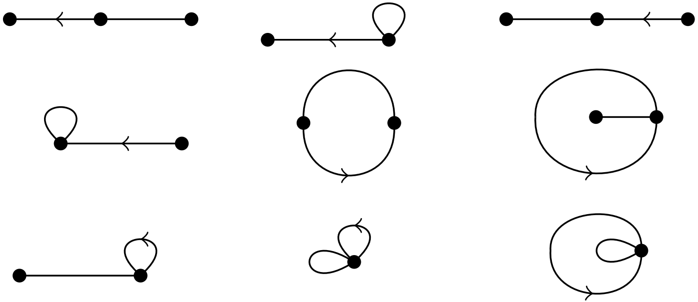
- A census of planar maps. *Canadian Journal of Mathematics*, 15:249–271, 1963.
- On the enumeration of planar maps. *Bulletin of the American Mathematical Society*, 74:64–74, 1968.

•

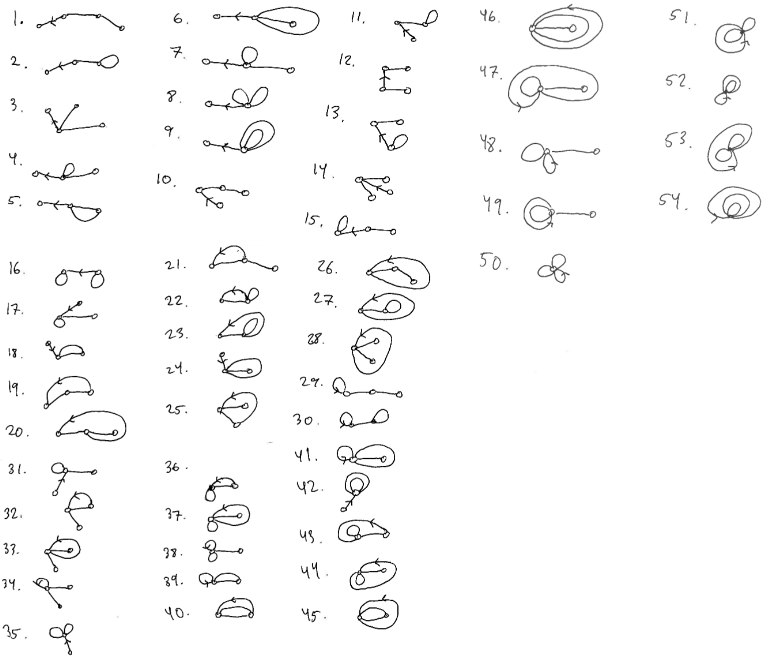
1



2



9


1.
2.
3.
4.
5.
6.
7.
8.
9.
10.
11.
12.
13.
14.
15.
16.
17.
18.
19.
20.
21.
22.
23.
24.
25.
26.
27.
28.
29.
30.
31.
32.
33.
34.
35.
36.
37.
38.
39.
40.
41.
42.
43.
44.
45.
46.
47.
48.
49.
50.
51.
52.
53.
54.

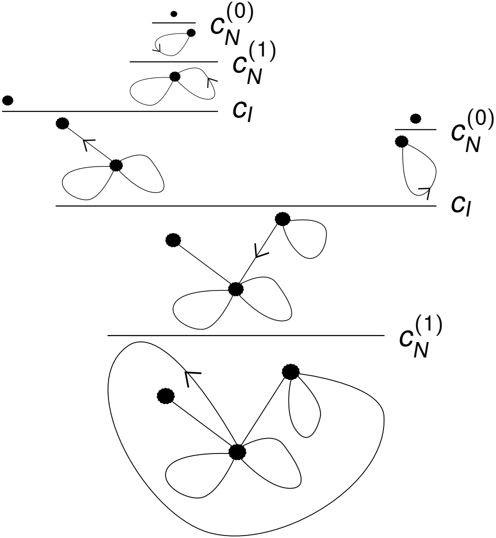
Met Alain Giorgetti at MAP(!) 2014 workshop in Paris.

We wrote a paper:

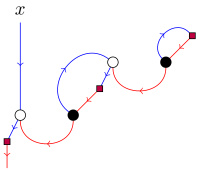
- A correspondence between rooted planar maps and normal planar lambda terms. August 21, 2014. arxiv.org/abs/1408.5028

Idea: replay **Tutte decomposition** in lambda calculus.

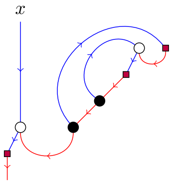
The proof is presented using *string diagrams*.



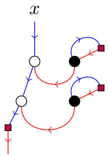
$c_N^{(0)}$
$c_N^{(1)}$
c_l
$c_N^{(0)}$
c_l
$c_N^{(1)}$



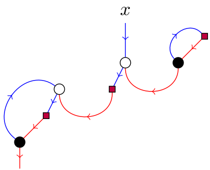
$x(\lambda y.y(\lambda z.z))$



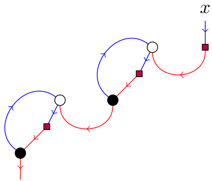
$x(\lambda y.\lambda z.zy)$



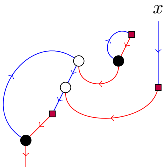
$(x(\lambda y.y))(\lambda z.z)$



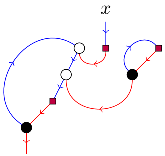
$\lambda y.y(x(\lambda z.z))$



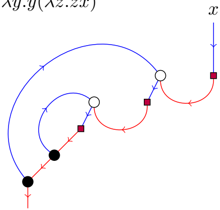
$\lambda y.y(\lambda z.zx)$



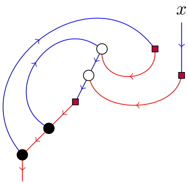
$\lambda y.(y(\lambda z.z))x$



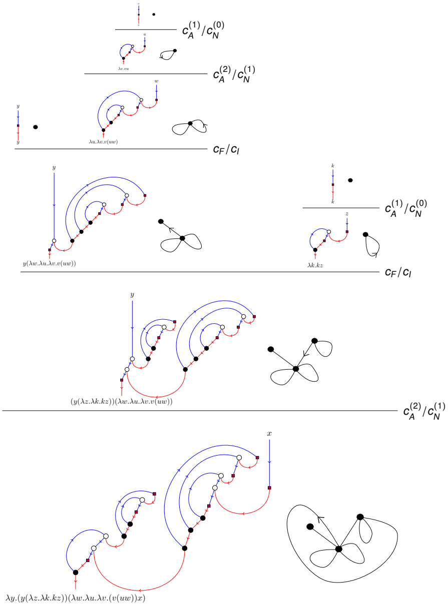
$\lambda y.(yx)(\lambda z.z)$



$\lambda y.\lambda z.z(yx)$



$\lambda y.\lambda z.(zy)x$

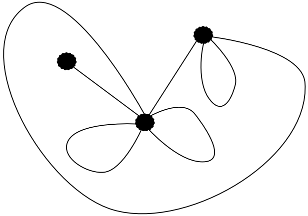

c_A^(1)/c_N^(0)
λv.vu
c_A^(2)/c_N^(1)
λu.λv.v(uw)
c_F/c_I
y(λw.λu.λv.v(uw))
c_A^(1)/c_N^(0)
λk.kz
c_F/c_I
(y(λz.λk.kz))(λw.λu.λv.v(uw))
c_A^(2)/c_N^(1)
λy.(y(λz.λk.kz))(λw.λu.λv.(v(uw))x)

Does all this mean anything?

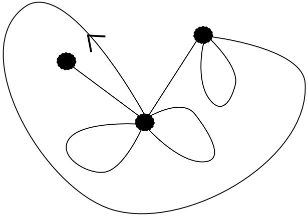
Well, it's not completely clear.

But it raises tantalizing questions in both directions...

From maps to lambda calculus



A planar map

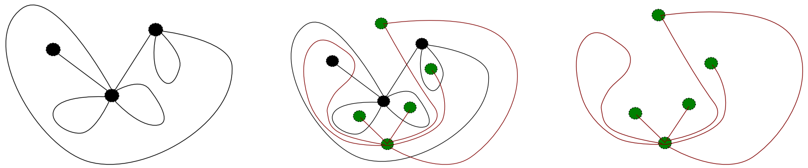


A rooted planar map

Tutte originally considered rooted maps because they were easier to count than unrooted maps, which can have non-trivial symmetries.

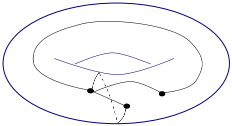
What (if anything) does it mean to “unroot” a lambda term?

Swapping faces with vertices is a dualizing operation on maps:



What (if anything) is the meaning of face-vertex duality in lambda calculus?

In general, a map need not be planar—one can consider graphs embedded on surfaces of arbitrary genus, for example on a torus:



Is there a natural notion of genus for lambda terms?

From lambda calculus to maps

The bijection is between rooted planar maps and normal planar lambda terms, but of course the main interest of lambda calculus is that we can *compute* with terms, i.e., reduce an arbitrary term to normal form.

What (if anything) is the process for which rooted maps are normal forms?

Every linear lambda term has a principal type that uniquely identifies its normal form, and, in general, types enable various operations (*product*, *implication*, etc.) and relations (*entailment*, *isomorphism*, etc.).

What (if anything) do types tell us about maps?

Just as there is a dualizing operation on maps that swaps vertices and faces, in programming there is a natural notion of computational duality between *values* and *continuations*.

What (if anything) is the meaning of computational duality for maps?